Combinatorics & Card Tricks

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The Gilbert–Shannon–Reeds model for riffle shuffling cards:

Deck of n cards, e.g. $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \times \{\mathrm{A}, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{T}, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$

CUT the cards	**DROP** the cards
“about in half”	“from larger pile”
binomial distribution	proportional to size

$$P(\text{cut } c \text{ cards deep}) = \frac{1}{2^n}\binom{n}{c} \qquad P(\text{drop from } L) = \frac{{}^{\#}L}{{}^{\#}L + {}^{\#}R}$$

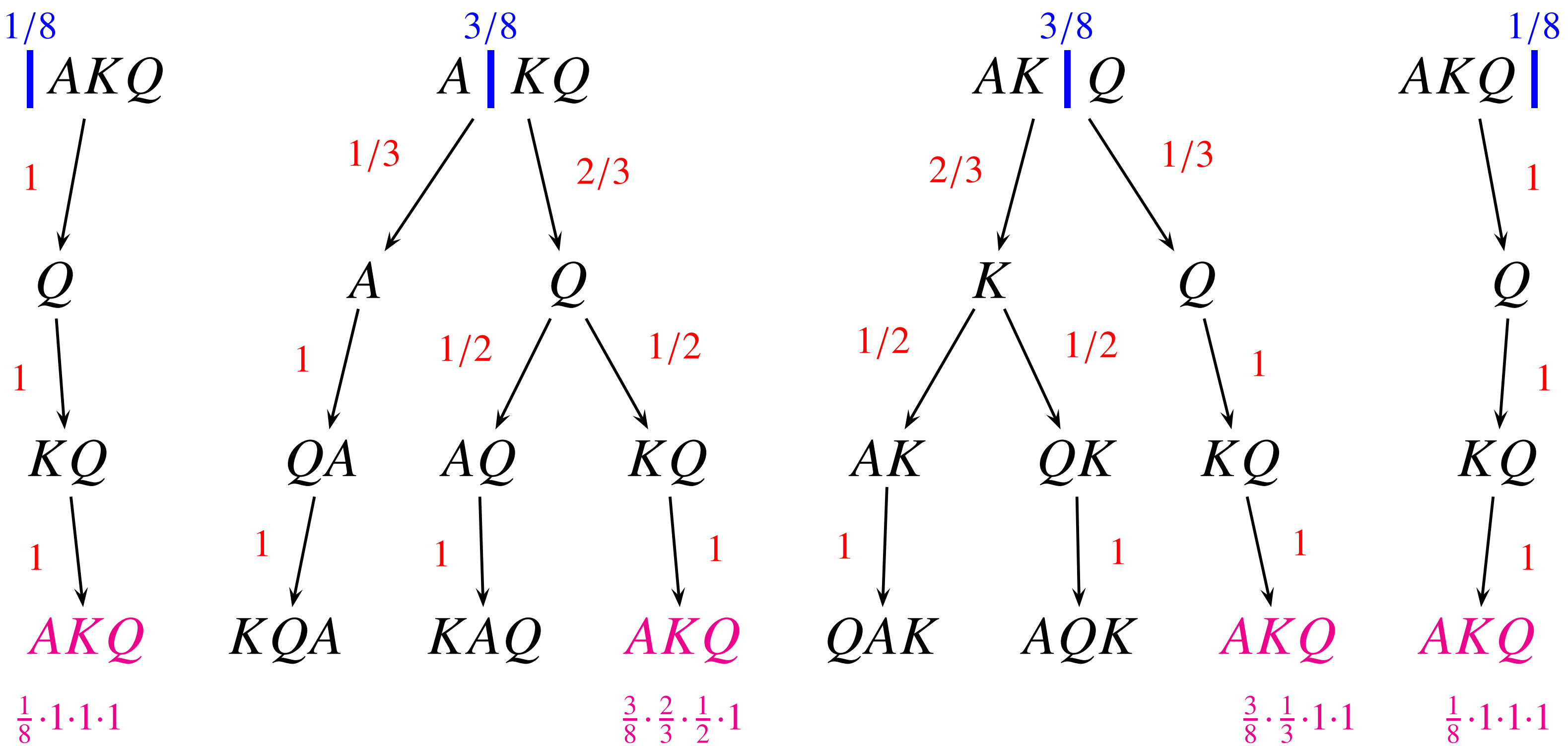
This defines a probability distribution Q on permutations of the deck:

Let $Q(\sigma)$ be chance that σ results from one riffle shuffle of the deck.

Repeated shuffles are defined by convolution powers:

$$Q^{*k}(\sigma) = \sum_{\tau} Q(\tau) Q^{*(k-1)}(\sigma\tau^{-1})$$

Shuffling three cards



σ	AKQ	AQK	QAK	KQA	KAQ	QKA
$Q(\sigma)$	$1/2$	$1/8$	$1/8$	$1/8$	$1/8$	0

Question: Why does QKA never result from a single shuffle of AKQ?

Answer: Each shuffle *doubles* the number of *rising sequences*:

4 1 5 8 2 9 6 7 3

A single riffle shuffle of AKQ will have at most two rising sequences, but QKA has three: One shuffle isn't enough to get from AKQ to QKA.

Question: How does Gordon's "Premo" card trick work?

Answer: After 3 riffle shuffles, there are $2^3 = 8$ rising sequences.
Each rising sequence has average length $52/8 = 6.5$.
Moving the top card creates a 9th rising sequence of size 1.
This short rising sequence is the card!

Disclaimer: This trick has an 84% chance of success.

Distribution after a single shuffle

Let $Q_2(\sigma)$ be chance that σ results from a riffle shuffle of the deck.
Let U be the uniform distribution, e.g. $U(\sigma) = \frac{1}{52!}$ for a standard deck.

σ	AKQ	AQK	QAK	KQA	KAQ	QKA
$Q_2(\sigma)$	$1/2$	$1/8$	$1/8$	$1/8$	$1/8$	0
$U(\sigma)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

There are several notions of the distance between Q_2 and U:

$$\|Q_2 - U\|_{TV} = \frac{1}{2}\sum_{\sigma \in \mathcal{S}_n} |Q_2(\sigma) - U(\sigma)| = \frac{1}{2}\left(\frac{1}{3} + 4\frac{1}{24} + \frac{1}{6}\right) = \frac{1}{3}$$

$$\mathrm{sep} = \max_{\sigma \in \mathcal{S}_n} 1 - \frac{Q_2(\sigma)}{U(\sigma)} = \max\{-2, \tfrac{1}{4}, 1\} = 1$$

Separation bounds total variation: $0 \le \|\mathbf{Q_2} - \mathbf{U}\|_{\mathbf{TV}} \le \mathrm{sep}(\mathbf{k}) \le 1$

Repeated riffle shuffles

Repeated shuffles are defined by convolution powers

$$Q_2^{*k}(\sigma) = \sum_{\tau} Q_2(\tau) Q_2^{*(k-1)}(\sigma\tau^{-1})$$

For Q_2^{*2}, for each of the $n!$ configurations, compute 2^n possibilities.

An a-shuffle is where the deck is cut into a packets with multinomial distribution and cards are dropped proportional to packet size.

CUT with probability

$$\frac{1}{a^n}\binom{n}{c_1, c_2, \ldots, c_a}$$

DROP proportional to size

$$\frac{{}^{\#}H_i}{{}^{\#}H_1 + {}^{\#}H_2 + \ldots + {}^{\#}H_a}$$

Let $Q_a(\sigma)$ be chance that σ results from an a-shuffle of the deck.

Theorem(Bayer–Diaconis) For any a, b, we have $Q_a * Q_b = Q_{ab}$

How many shuffles is enough?

Theorem (Bayer–Diaconis) Let r be the number of rising sequences.

$$\mathbf{Q_a(\sigma) = \frac{1}{a^n}\binom{n+a-r}{n}}$$

Proof: Given a cut, each σ that can result is equally likely, so we just need to count the number of cuts that can result in σ.

Classical stars (★) and bars (|) with n ★'s and $a-1$ | 's of which $r-1$ are fixed. So $n+a-r$ spots and choose n spots for the ★'s. □

★ ★ ★ | ★ ★ ★ ★ | ★ ★

	1	2	3	4	5	6	7	8	9	10	11	12
TV	1.00	1.00	1.00	1.00	.924	.614	.334	.167	.085	.044	.021	.010
sep	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.996	.931	.732	.479	.278

Encoding information

Encode the cards with binary sequences, i.e. strings of 0's and 1's.

2 bits encode
$2^2 = 4$ suits

0	0	♣
0	1	♠
1	0	♦
1	1	♥

3 bits encode
$2^3 = 8$ ranks

0	0	0	8
0	0	1	A
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7

Encode people by: **Sitting** $= \mathbf{0}$ and **Standing** $= \mathbf{1}$.

This (almost) explains the first card, but what about the rest?

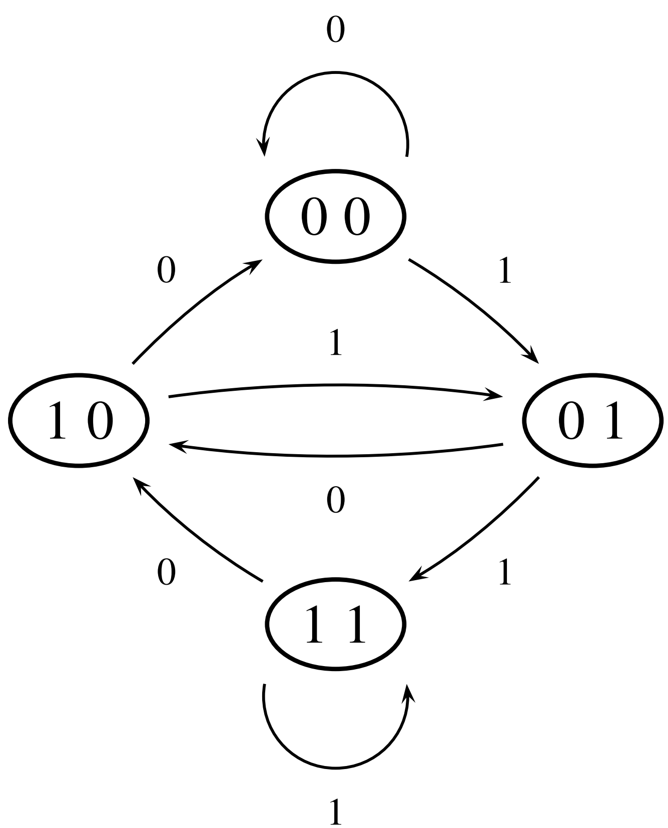
A de Bruijn sequence of order n is a cyclic sequence of length 2^n such that every subsequence of length n is unique.

Example. For $n = 3$, there are two: $0\ 0\ 0\ 1\ 0\ 1\ 1\ 1$ and $0\ 0\ 0\ 1\ 1\ 1\ 0\ 1$.

The de Bruijn Graph:

- nodes are unique binary sequences of length $n - 1$;
- directed edges given by $u \xrightarrow{b} v$ if $u_2 \cdots u_{n-1} b = v$.

Theorem. A de Bruijn sequence is an Eulerian cycle in the de Bruijn graph: cross each edge exactly once.



Some de Bruijn sequences have recurrence relations:

Example. For $n = 5$, one such sequence is

$$b_{k+5} = \begin{cases} 0 & \text{if } b_k = 1 \text{ and } b_{k+1} = b_{k+2} = b_{k+3} = b_{k+4} = 0 \\ 1 & \text{if } b_k = b_{k+1} = b_{k+2} = b_{k+3} = b_{k+4} = 0 \\ b_k + b_{k+3} \bmod 2 & \text{otherwise} \end{cases}$$

The resulting order 5 de Bruijn sequence is:

$$0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1$$

Using the encoding algorithm for suits and ranks, the deck order is:

8♣ A♣ 2♣ 5♣ 2♠ 5♦ 3♠ 7♦ 6♠ 5♥ 3♥ 6♦ 4♠ 8♥ A♦ 3♣
7♣ 7♠ 7♥ 6♥ 4♥ A♥ 3♦ 6♣ 5♠ 2♥ 4♦ A♠ 2♦ 4♣ 8♠ 8♦

References

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- S. Assaf, P. Diaconis and K. Soundararajan. A rule of thumb for riffle shuffles. *Annals of Applied Probability*, 2011.
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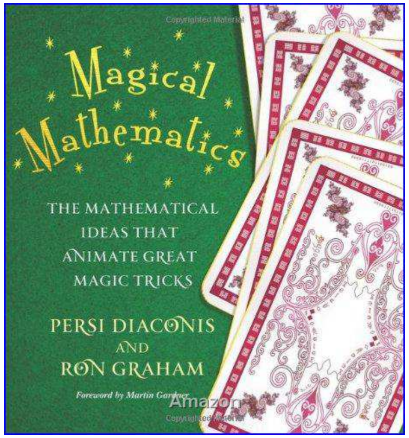

Thank You