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Real-time Bayesian data assimilation with data-based model enrichment for the monitoring of damage in materials and structures

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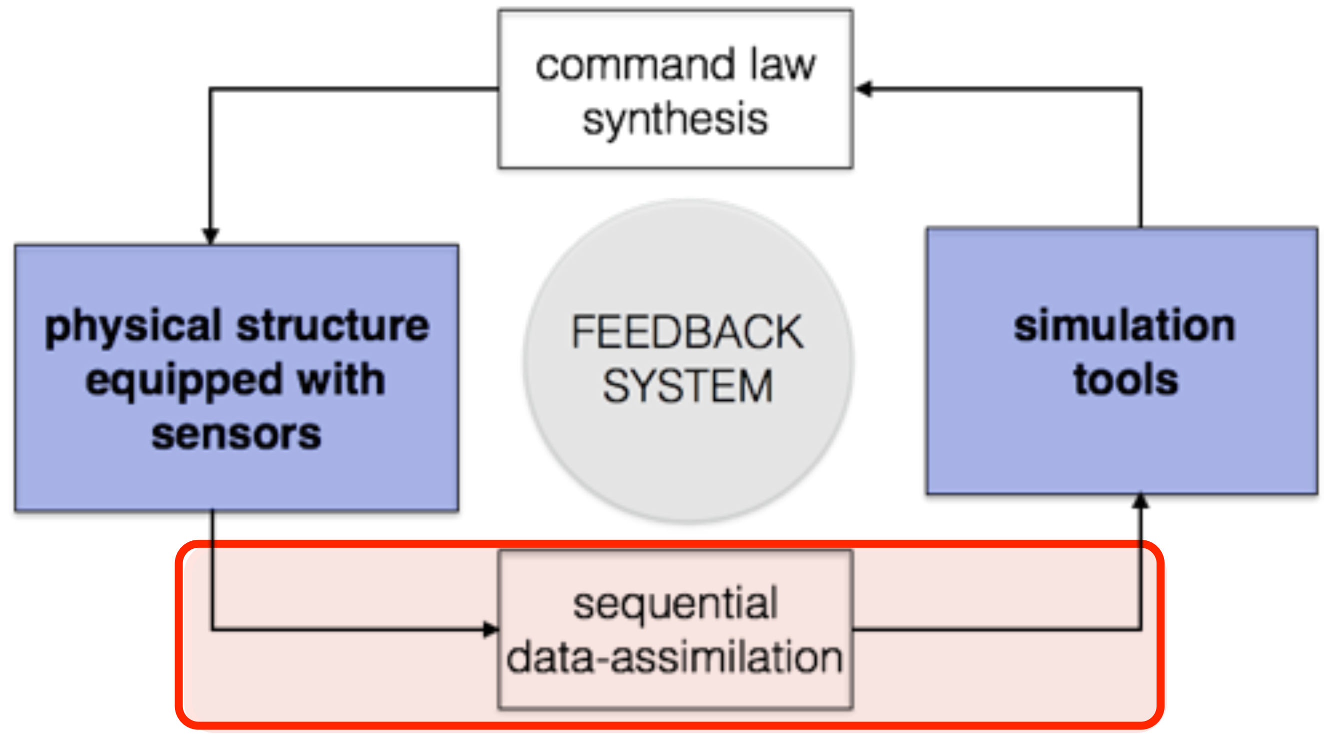
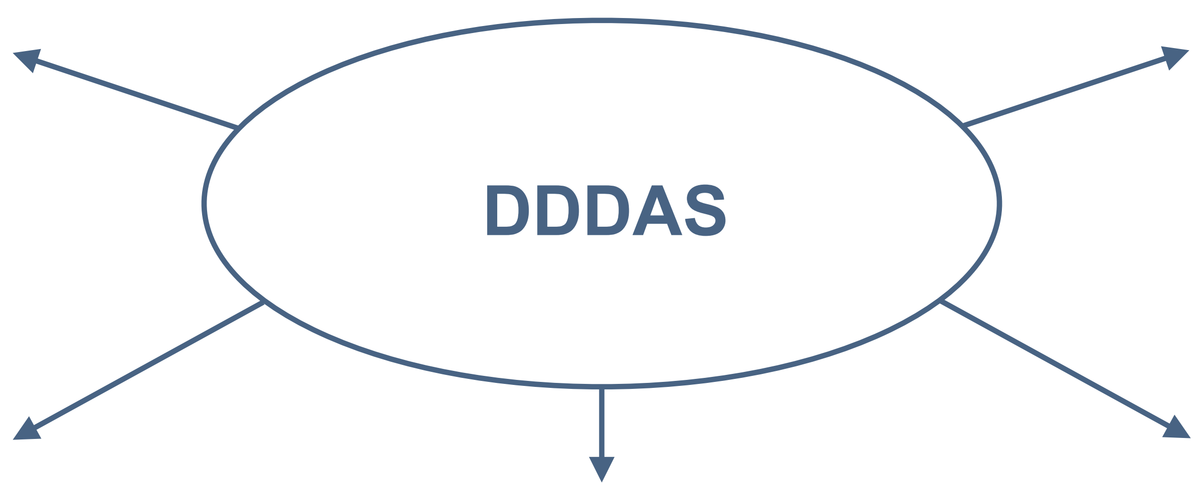
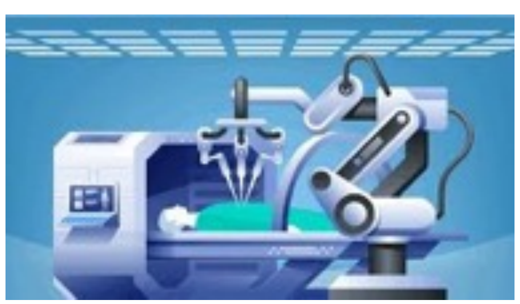
Context

Dynamic Data-Driven Application Systems (DDDAS) [Darema 04]



connected systems, cyber-physics,…

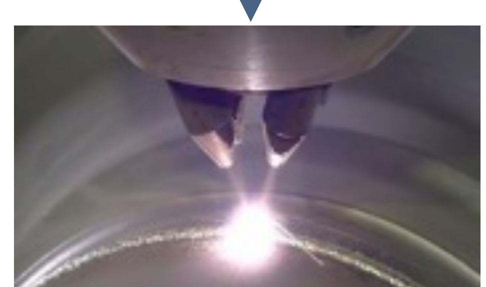




assisted surgery

renewable energy

building performance



manufacturing

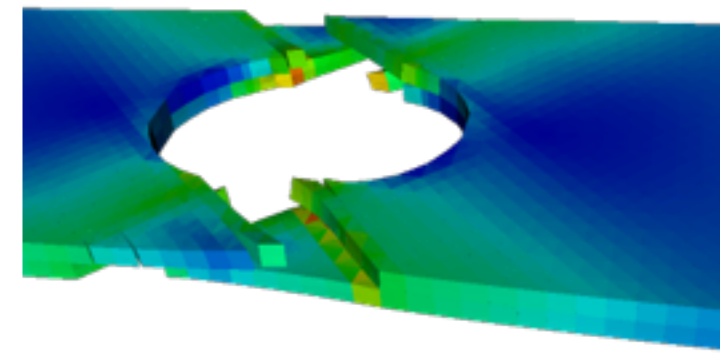
transportation

Context

- Application to **engineering structures** → **safety / reliability (degraded mode)**

embedded strain sensors
(defects ∼ 1mm)

complex physics



predictive simulations

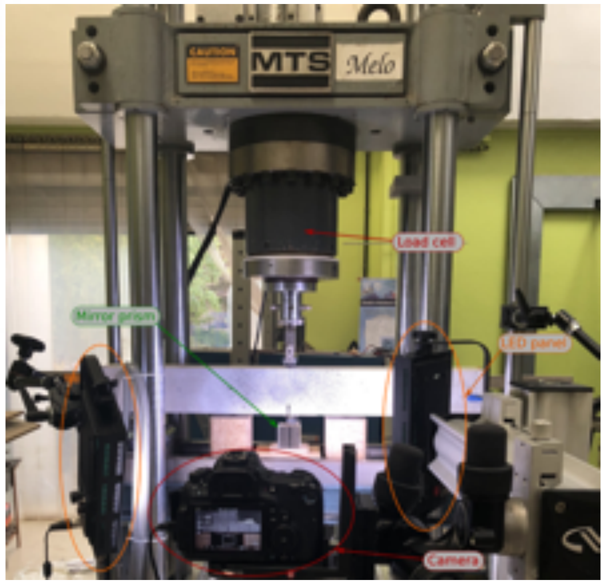
exploit the best of data & material models

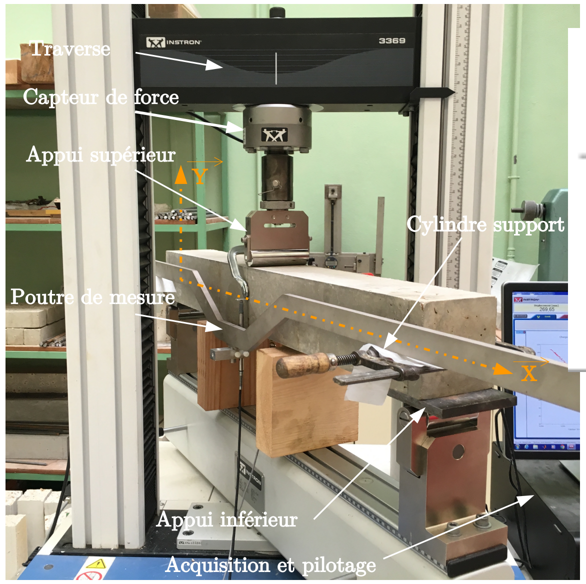
<u>GOAL</u>: **dynamical model updating** from data assimilated on-the-fly

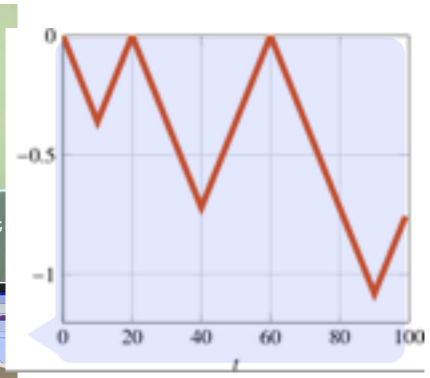
- for **complex nonlinear** large systems with **uncertain environment**
- in **real-time**
- from numerous, indirect and **noisy data** (images, optic fibers…)

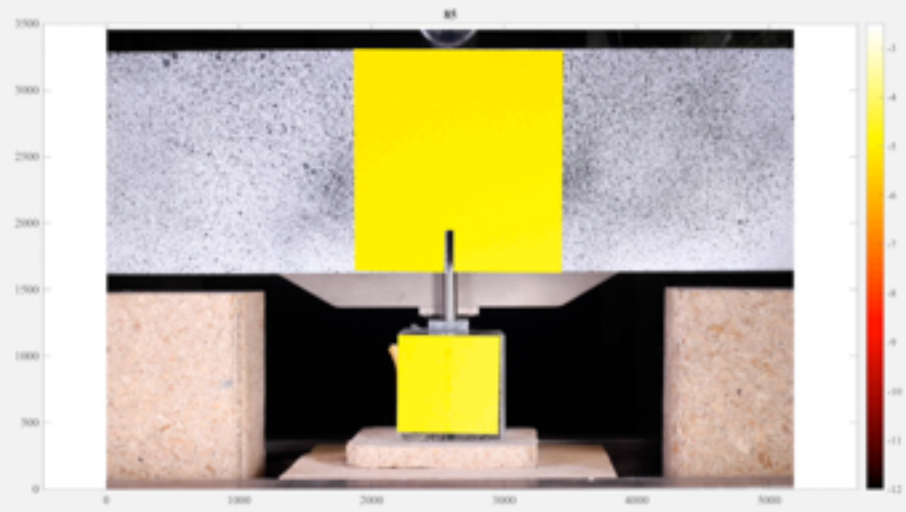
Application of interest

→ structural integrity on a large-scale damageable concrete structure





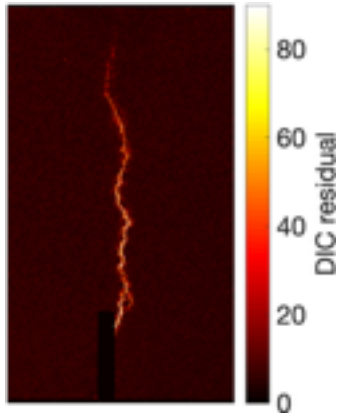




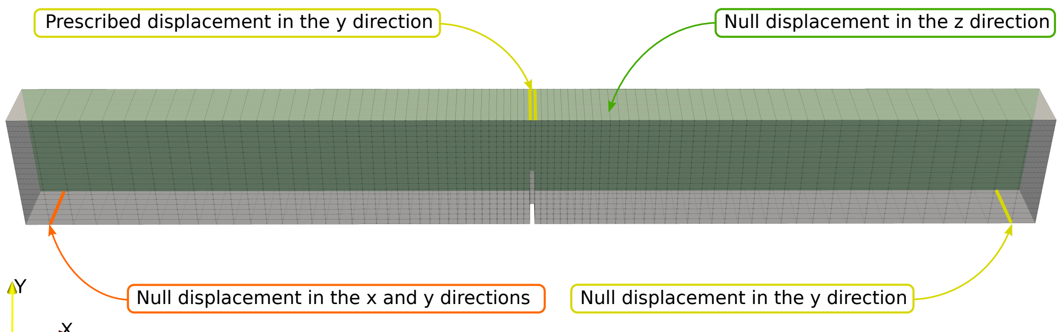
Ditical Image Correlation (DIC)

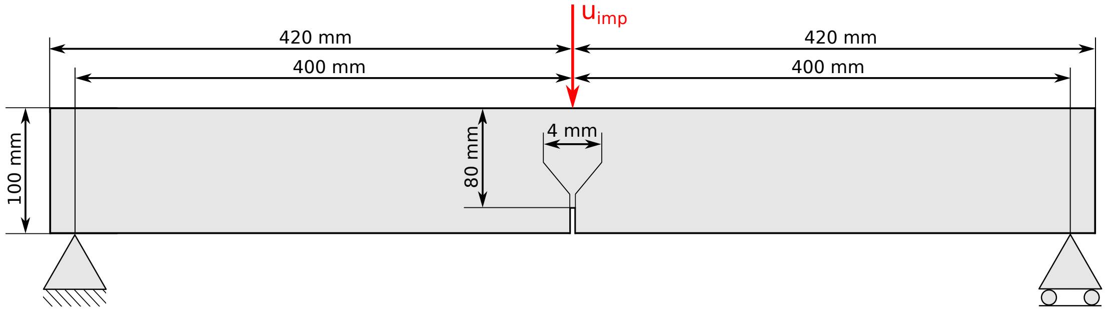
→ DIC pictures taken every 5s, and post-processed with Corelli [Leclerc *et al*. 15]

→ prediction of crack propagation & failure (before the physics!!)



→ simulator using an isotropic damage model





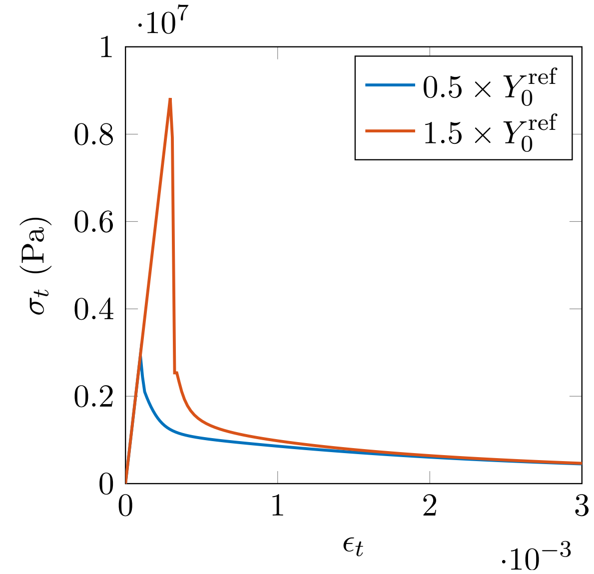
Objectives

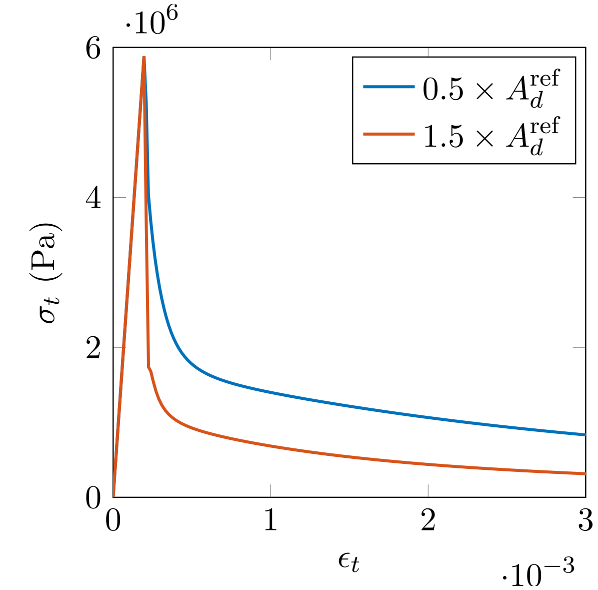
$$\boxed{\sigma = (1-d)\mathbf{C}\epsilon \quad ; \quad d(Y, A_d, Y_0) = 1 - \frac{1}{1 + A_d(Y - Y_0)}}$$

$Y = \frac{1}{2}\langle\epsilon\rangle_+ : \mathbf{C} : \langle\epsilon\rangle_+$ released energy rate

Y_0 initial threshold for damage initiation

A_d scalar brittleness (post-peak behavior)





⟶ **updating of parameters (Y_0, A_d) from data**

- robustness with uncertainties (measurement noise,...) ⟶ **Bayesian inference** [Kaipio & Sommersalo 04]
- real-time constraint ⟶ **Reduced Order Modeling (PGD)** [Chinesta *et al.* 14]
  ⟶ **Transport Map sampling** [El Moseley & Marzouk 12]
- model bias (BC, material (drying effects),...) ⟶ **online data-based correction**

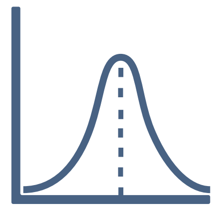
Outline

1. **Bayesian formulation with ROM**
2. **Real-time sampling using Transport Maps**
3. **Model bias correction from data**
4. **Feedback control**

Bayesian Formulation

- **Problem unknown**

$\boxed{\pi(\mathbf{p}|\mathbf{d}^{\mathrm{obs}})}$

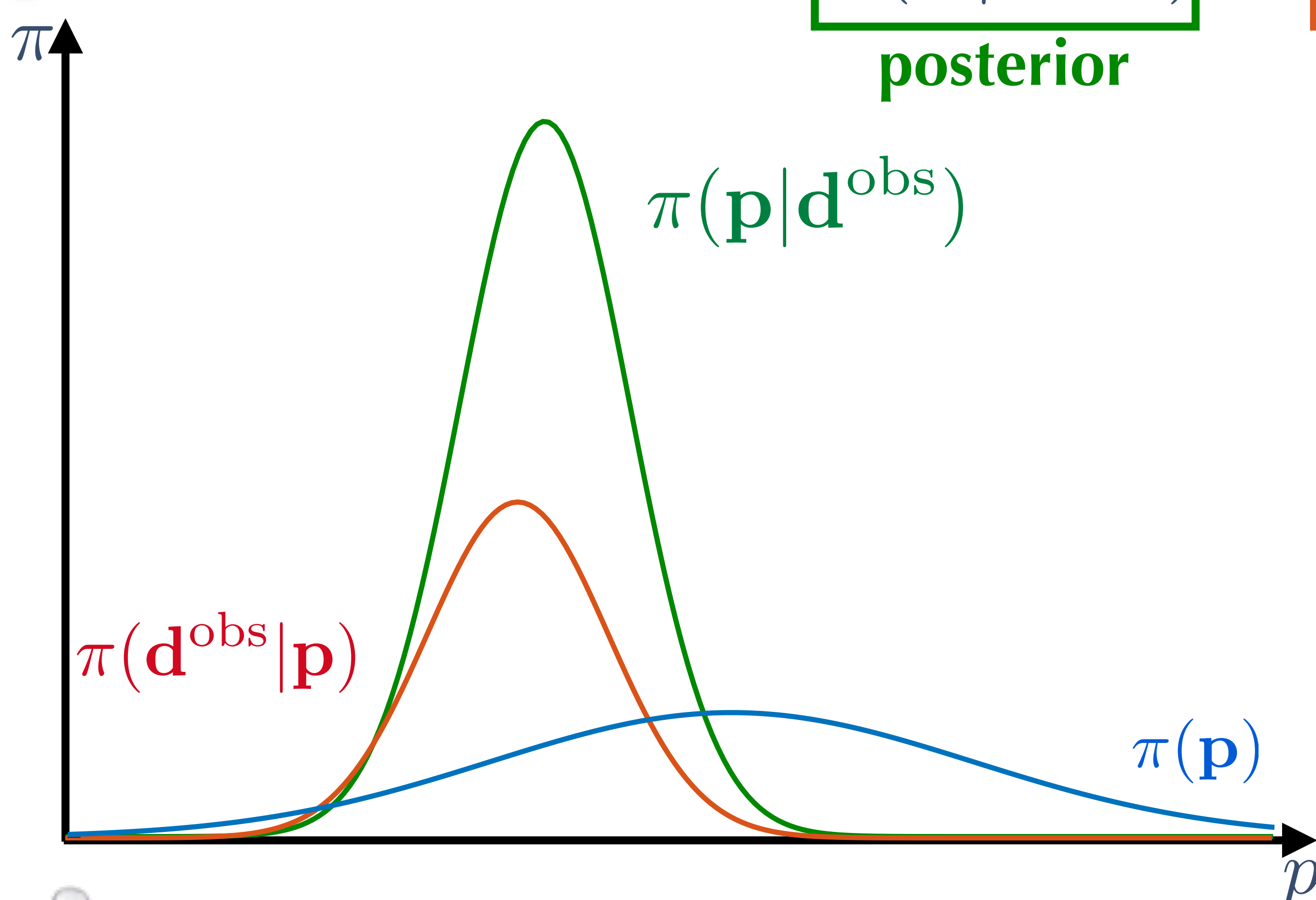


→ automatic regularization

→ natural framework to consider uncertainties

- **Bayes Theorem:**

$$\underbrace{\boxed{\pi(\mathbf{p}|\mathbf{d}^{\mathrm{obs}})}}_{\text{posterior}} \propto \underbrace{\boxed{\pi(\mathbf{d}^{\mathrm{obs}}|\mathbf{p})}}_{\text{likelihood}} . \underbrace{\boxed{\pi(\mathbf{p})}}_{\text{prior}}$$



$$= \pi_{\mathrm{meas}}(\mathbf{d}^{\mathrm{obs}} - \mathbf{d}(\mathbf{p}))$$

if additive measurement noise

costly multi-query process

- **Sequential assimilation**

$$\pi(\mathbf{p}|\mathbf{d}_1^{\mathrm{obs}}, ..., \mathbf{d}_i^{\mathrm{obs}}) \propto \left(\prod_{j=1}^{i} \pi_{t_j}(\mathbf{d}_j^{\mathrm{obs}}|\mathbf{p}) \right) .\pi_0(\mathbf{p}) \qquad \pi_{t_j}(\mathbf{d}_j^{\mathrm{obs}}|\mathbf{p}) = \pi_{\mathrm{meas}}\left(\mathbf{d}_j^{\mathrm{obs}} - \mathbf{d}\left(\mathbf{p}, t_j\right)\right)$$

PGD Model Reduction

- modal description of multiparametric solution [Nouy 10, Chinesta *et al.* 14]
- low-rank canonical tensor format (separated variables)

$$\boxed{\mathbf{u}_m(\mathbf{x}, t, \mathbf{p}) = \sum_{k=1}^{m} \boldsymbol{\Lambda}_k(\mathbf{x})\lambda_k(t) \prod_{i=1}^{d} \alpha_k^i(p_i)}$$

⟶ construction in the offline phase

⟶ explicit dependency on parameters

⟶ computed using the LATIN-PGD algorithm for nonlinear models[Vitse *et al.* 19]

⟶ straightforward model evaluation in the likelihood function

$$\mathbf{d}(\mathbf{p}, t) = \mathcal{O}\left(\mathbf{u}_m(\mathbf{x}, t, \mathbf{p})\right)$$ [Berger *et al.* 17, Rubio *et al.* 18]

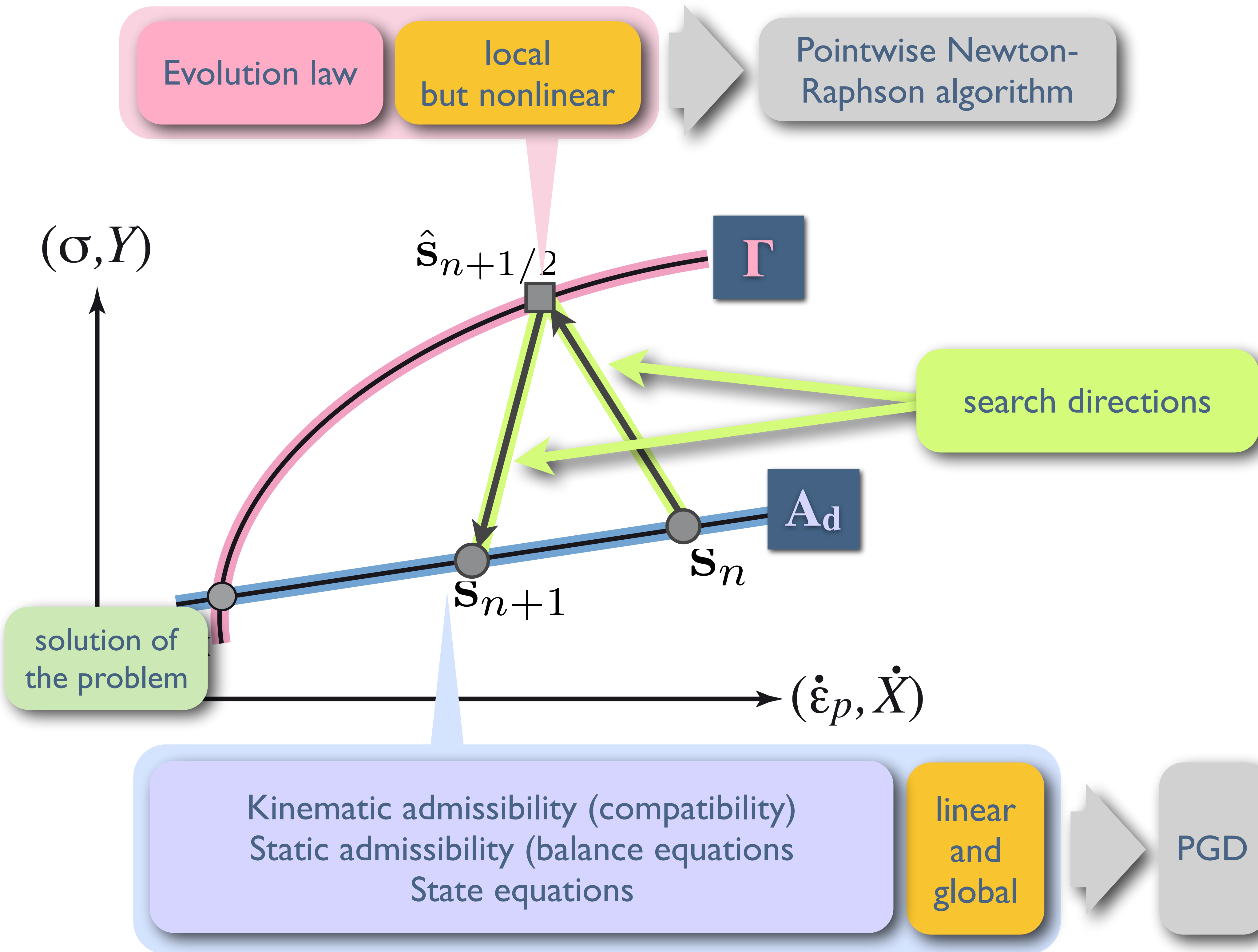
⟶ fast UQ on outputs of interest $q(\mathbf{p}) \approx \mathcal{Q}\left(\mathbf{u}_m(\mathbf{x}, t, \mathbf{p})\right)$

⟶ samples $q_k = q(\mathbf{p}_k)$

Extension to NL problems

Use of the iterative LATIN-PGD algorithm [Ladevèze *et al.* 10]

→ *non-incremental iterative strategy*



PGD Modes

$$\boxed{\mathbf{u}_m(\mathbf{x}, t, Y_0, A_d) = \sum_{k=1}^{m} \mathbf{\Lambda}_k(\mathbf{x})\lambda_k(t)\alpha_k^1(Y_0)\alpha_k^2(A_d)}$$

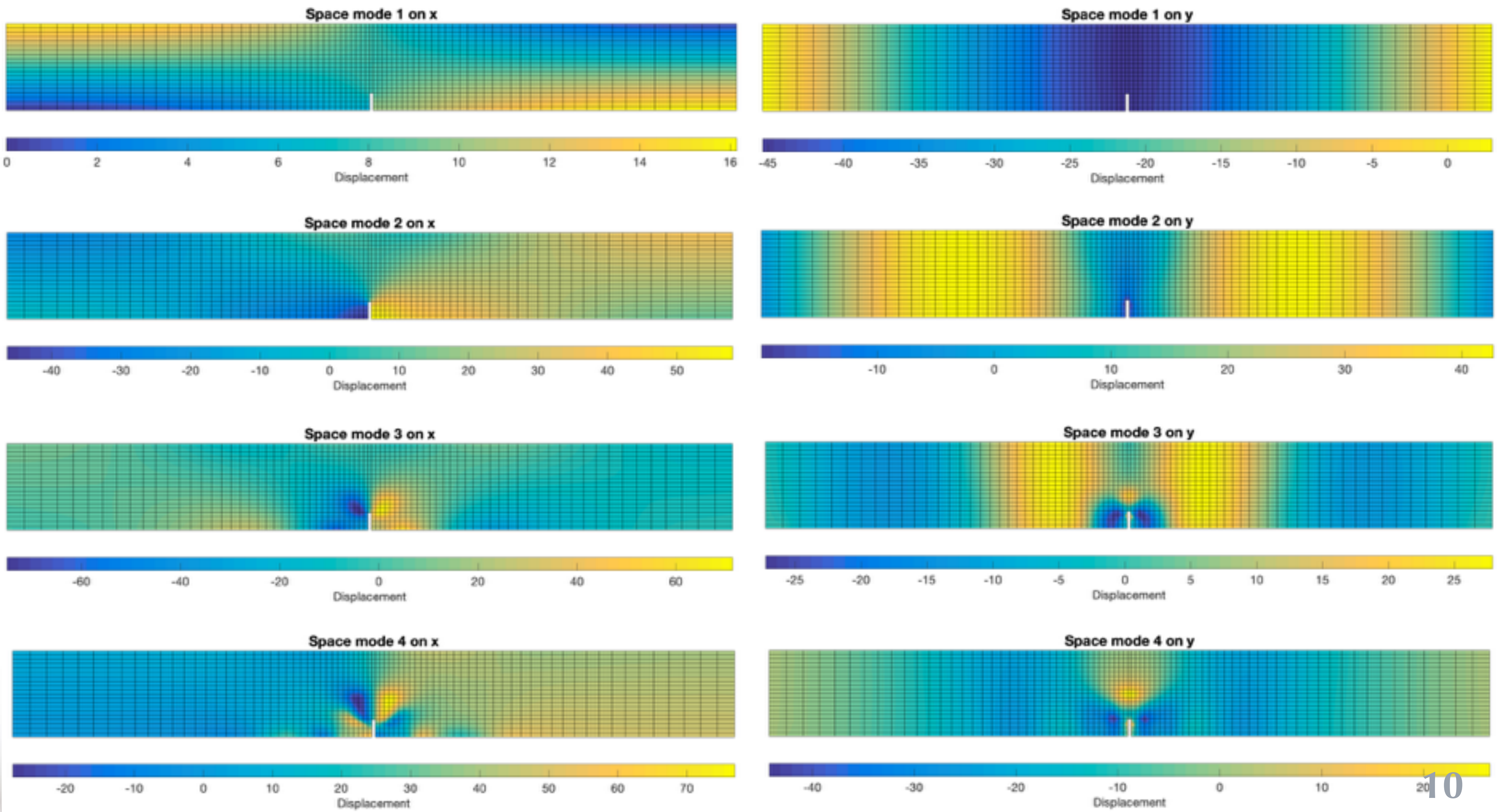
100 loading steps

$E = 30\ GPa \qquad \nu = 0.23$

$Y_0^{\mathrm{ref}} = 216\ J.m^{-3}$

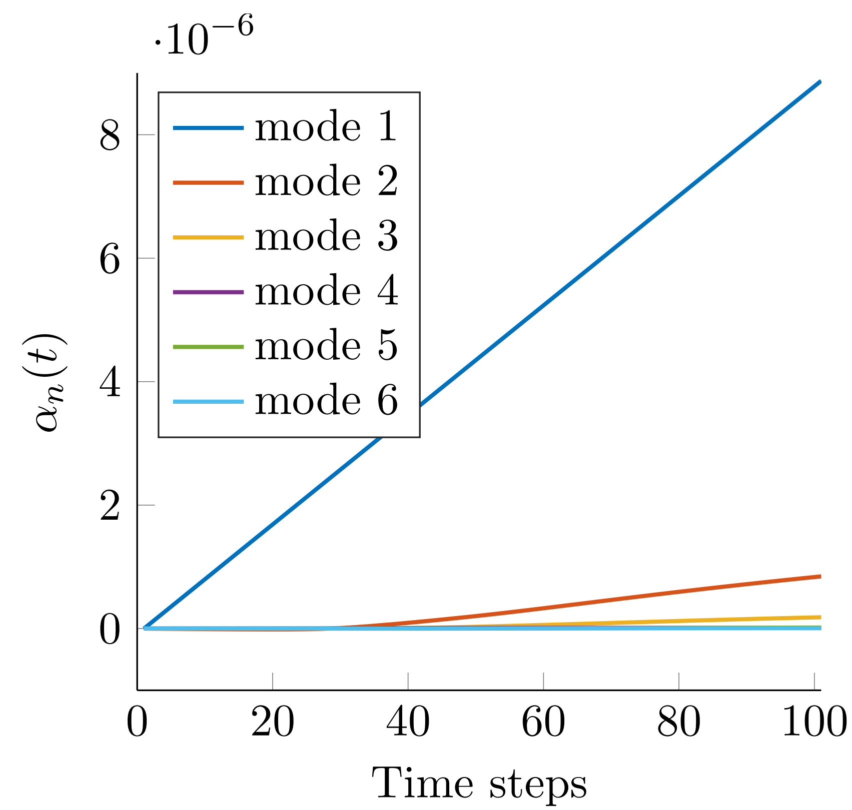
$A_d^{\mathrm{ref}} = 2.25\ J^{-1}.m^3$

- Space modes $\mathbf{\Lambda}_k$

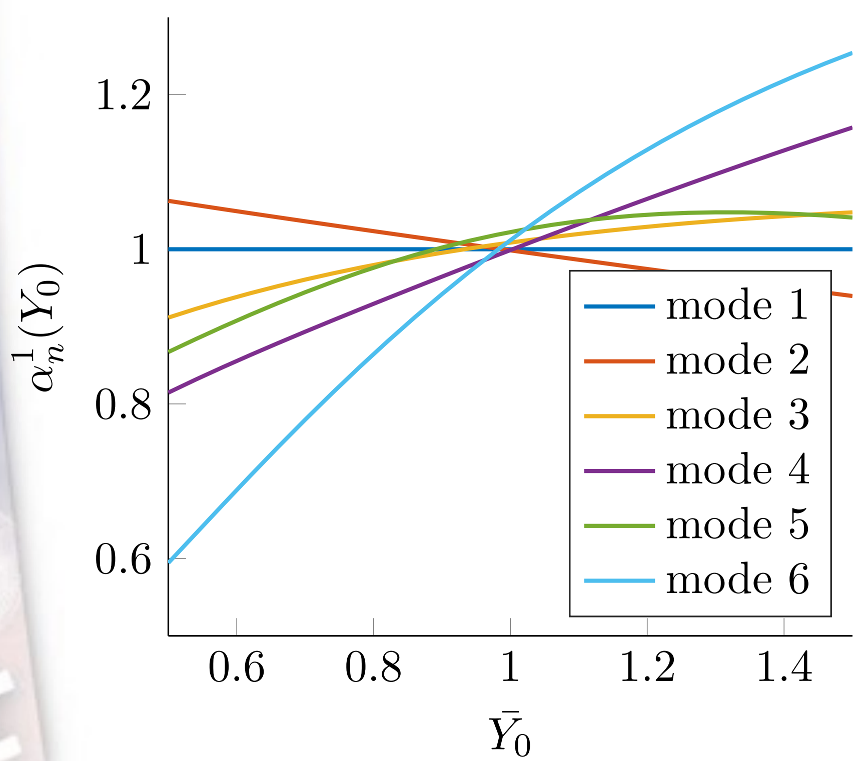


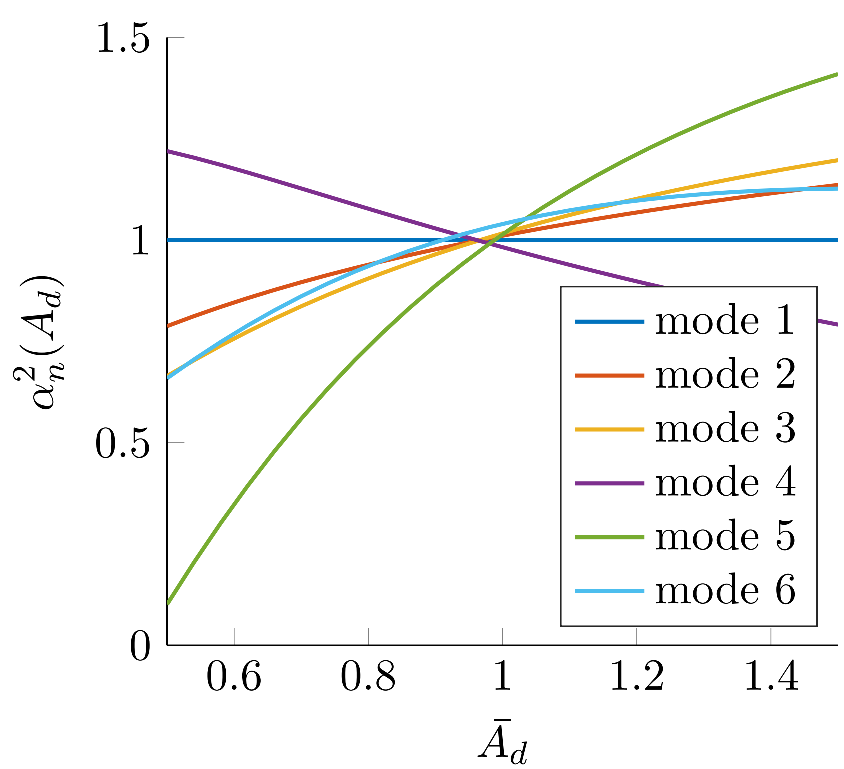
PGD Modes

- Time modes λ_k



- Parameter modes (α_k^1, α_k^2) (with $\bar{Y}_0 = Y_0 / Y_0^{\text{ref}}$, $\bar{A}_d = A_d / A_d^{\text{ref}}$)





1st mode

=

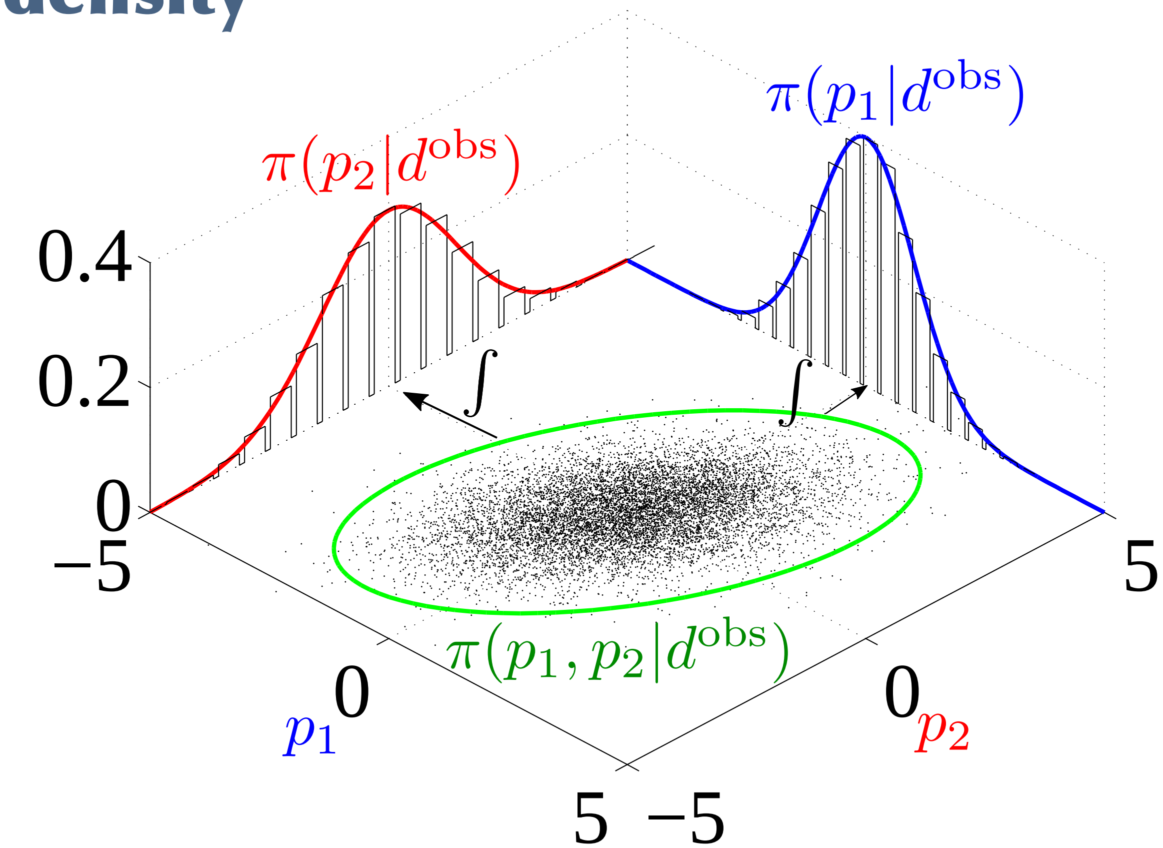
full elasticity

Sampling of Posterior PDFs

Characterization of the posterior density

- Mean *a posteriori*
- Maximum *a posteriori*
- 1D Marginals
- Uncertainty propagation

➡ Need multi-dimensional integration



Monte-Carlo integration:

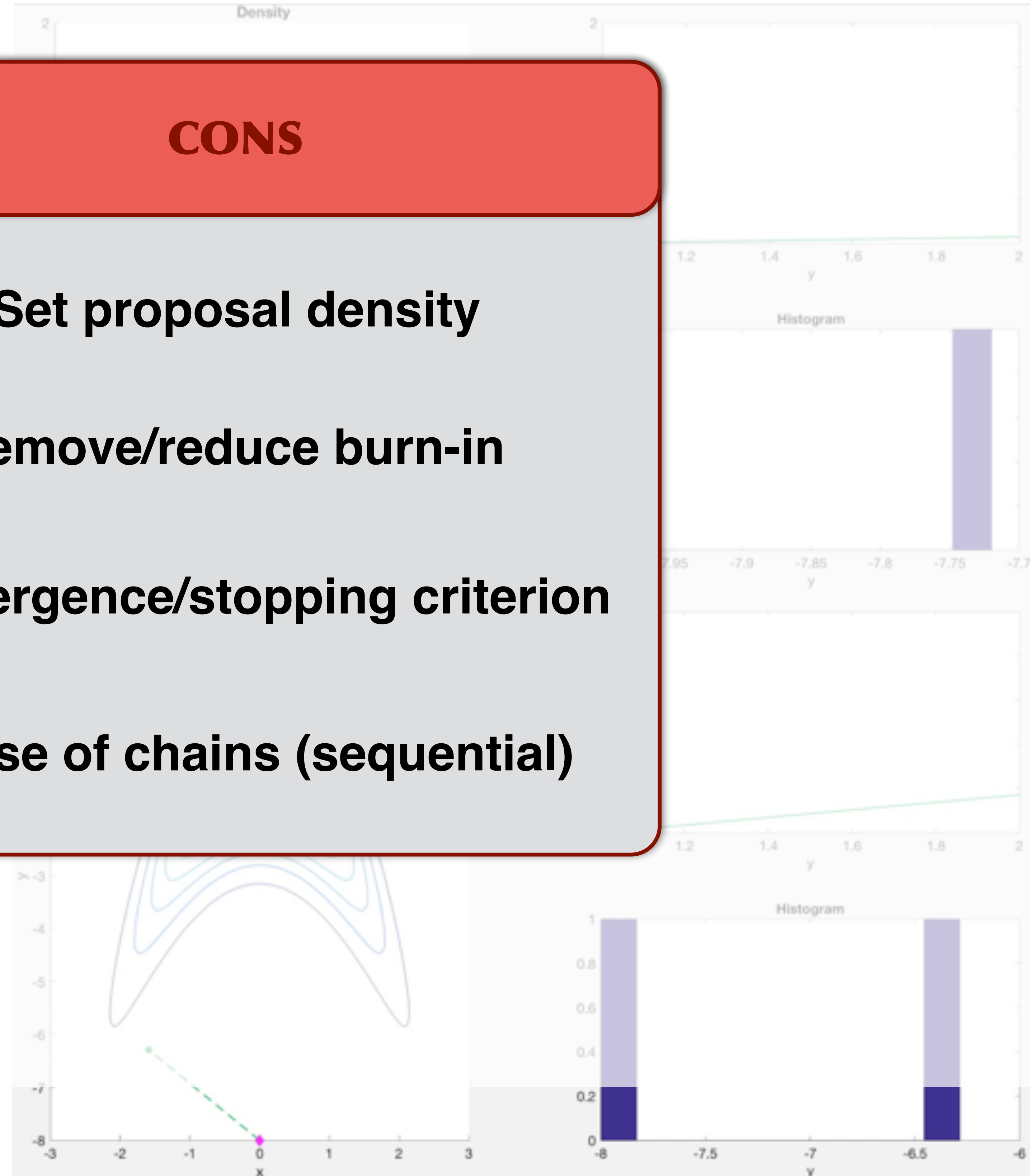
- Quantity of interest: $\mathbb{E}[h] = \int h(\mathbf{p})\pi(\mathbf{p})d\mathbf{p}$
- With samples $\mathbf{p}^{\{1,\ldots,N\}} \sim \pi : \mathbb{E}[h] \approx \overline{h} = \frac{1}{N}\sum_{i=1}^{N} h(\mathbf{p}^i)$

➡ Need samples from posterior density

Sampling of Posterior PDFs

Markov Chain Monte-Carlo (MCMC) method

- **small** proposal
- **large** proposal



CONS

Set proposal density

Remove/reduce burn-in

Convergence/stopping criterion

Reuse of chains (sequential)

Outline

1. Bayesian formulation with ROM
2. Real-time sampling using Transport Maps
3. Model bias correction from data
4. Feedback control

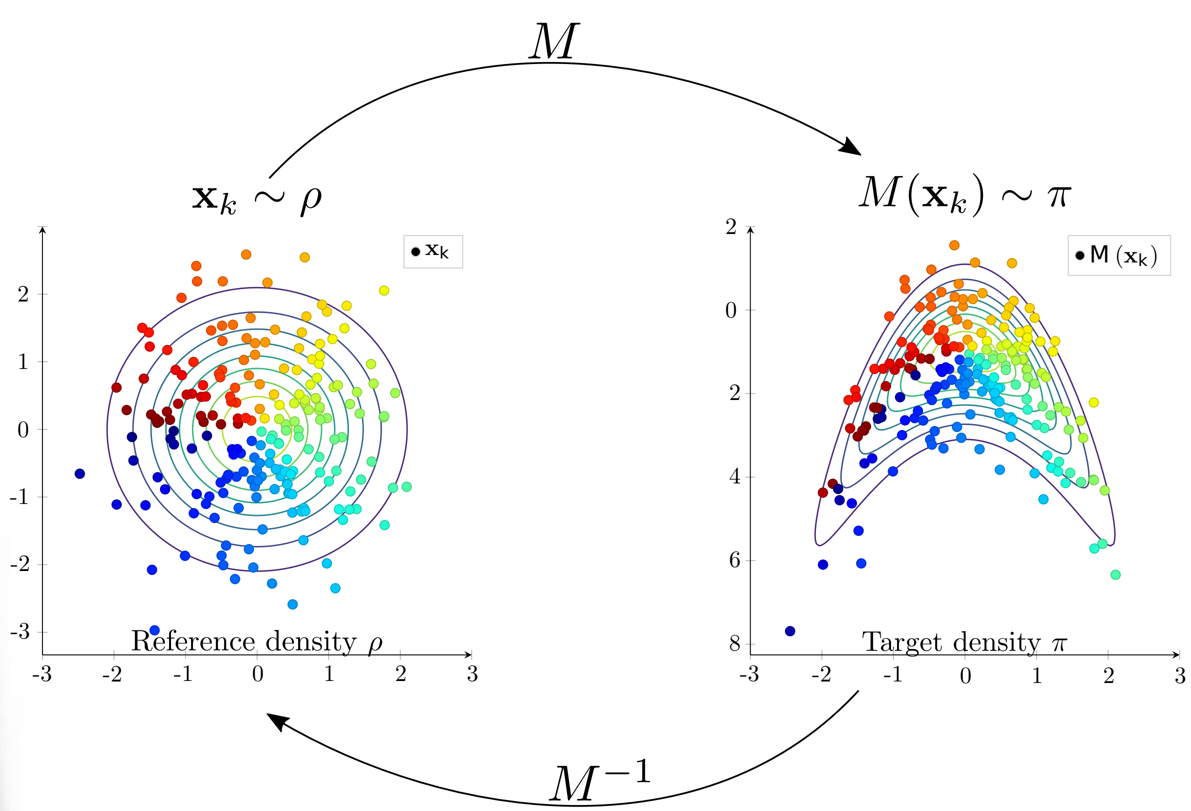
Transport Map Sampling

- **Transport Maps method** [Villani 07, El Moselhy & Marzouk 12]

Transport integrals over the **target** density to integrals over a **reference** density

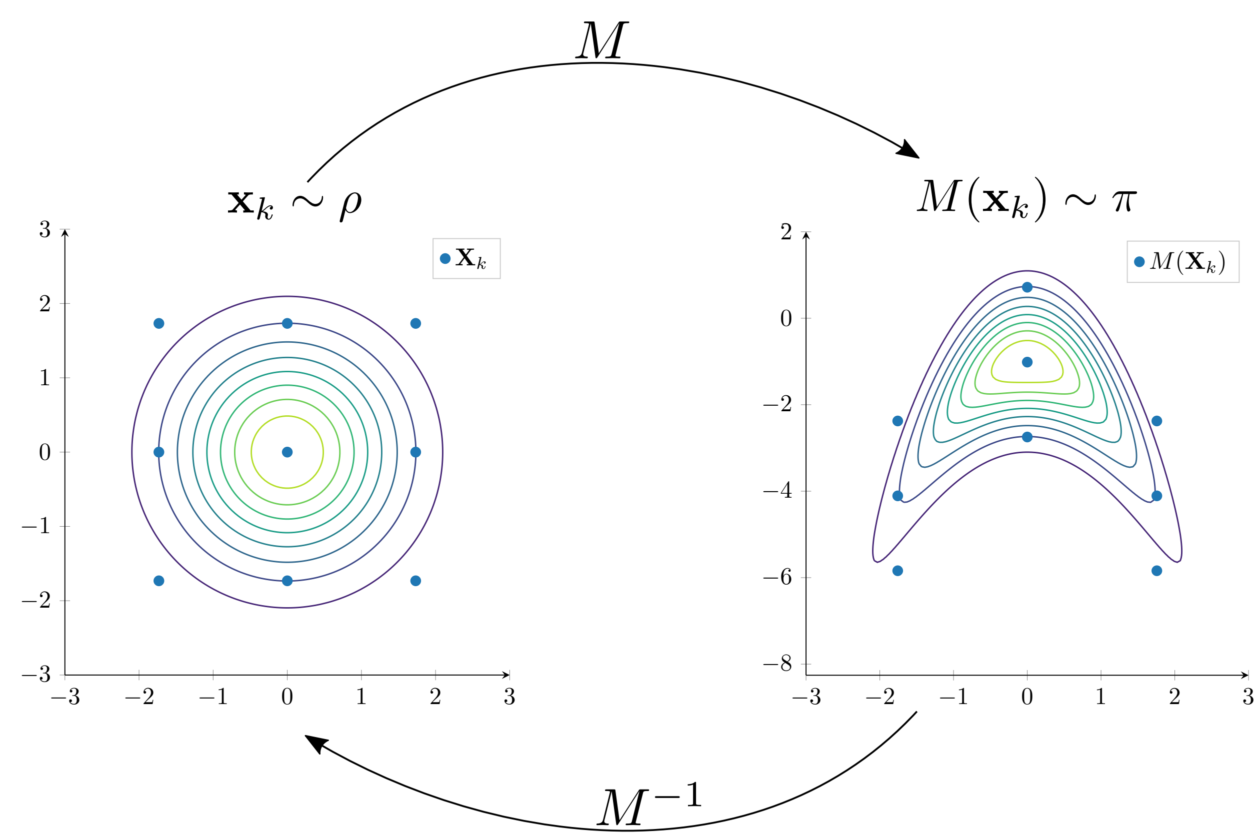
$$\boxed{\int g \, \mathrm{d}\nu_\pi = \int g \circ M \, \mathrm{d}\nu_\rho}$$

deterministic application



transport of samples

(pushing Gaussian samples through the map)



transport of quadrature rules

→ computations (sampling, integration) performed in the reference space [Marzouk 16]

Transport Map Sampling

Parametrization of Transport Maps

- **Structure:**

$$M(\mathbf{p}) = \begin{bmatrix} M^1(\mathbf{a}_c^1, \mathbf{a}_e^1, p_1) \\ M^2(\mathbf{a}_c^2, \mathbf{a}_e^2, p_1, p_2) \\ \vdots \\ M^d(\mathbf{a}_c^d, \mathbf{a}_e^d, p_1, p_2, ..., p_d) \end{bmatrix}$$

Knothe-Rosenblatt rearrangements (lower triangular monotonic maps)

- unique minimizer
- computational feasibility (inversion)
- optimality for a weighted metric [El Moselhy & Marzouk 12]

- **Parametrization:**

$$M^k(\mathbf{a}_c^k, \mathbf{a}_e^k, \mathbf{p}) = \Phi_c(\mathbf{p})\mathbf{a}_c^k + \int_0^{p_k} (\Phi_e(p_1, ..., p_{k-1}, \theta)\mathbf{a}_e^k)^2 \mathrm{d}\theta$$

Φ_c, Φ_e : Hermite polynomials which given order

$\mathbf{a}_c, \mathbf{a}_e$: parameters

→ obtained from miminization of Kullback-Liebler divergence

$$\mathcal{D}_{KL}(M_\sharp \nu_\rho || \nu_\pi) = \mathbb{E}_\rho \left[\log \frac{\nu_\rho}{M_\sharp^{-1} \nu_\pi} \right]$$

Transport Map Sampling

Minimization problem

$$\boxed{\min_{\mathbf{a}_c^{1,\dots,d},\mathbf{a}_e^{1,\dots,d}} \sum_{i=1}^{N} \omega_i \left[-\log(\tilde{\pi} \circ M(\mathbf{a}_c^{1,\dots,d}, \mathbf{a}_e^{1,\dots,d}, \mathbf{p}_i) - \log(|\det \nabla M(\mathbf{a}_c^{1,\dots,d}, \mathbf{a}_e^{1,\dots,d}, \mathbf{p}_i))|)\right]}$$

$\tilde{\pi}(\mathbf{p}|\mathbf{d}^{\text{obs}}) = \pi_{\text{meas}}(\mathbf{d}^{\text{obs}} - u_m(\mathbf{p}))\pi(\mathbf{p})$ (non-normalized pdf)

→ solved with gradient/Hessian information (BFGS,…)

→ partial derivatives explicitly recovered and stored in the **offline phase**

$$\frac{\partial^n \mathbf{u}_m}{\partial p_j^n}(\mathbf{x}, t, \mathbf{p}) = \sum_{k=1}^{m} \mathbf{\Lambda}_k(\mathbf{x})\lambda_k(t)\frac{\partial^n \alpha_k^j}{\partial p_j^n}(p_j)\prod_{\substack{i=1\\ i\neq j}}^{d} \alpha_k^i(p_i)$$

→ large speed-up for the computation of maps!!! [Rubio *et al.* 19]

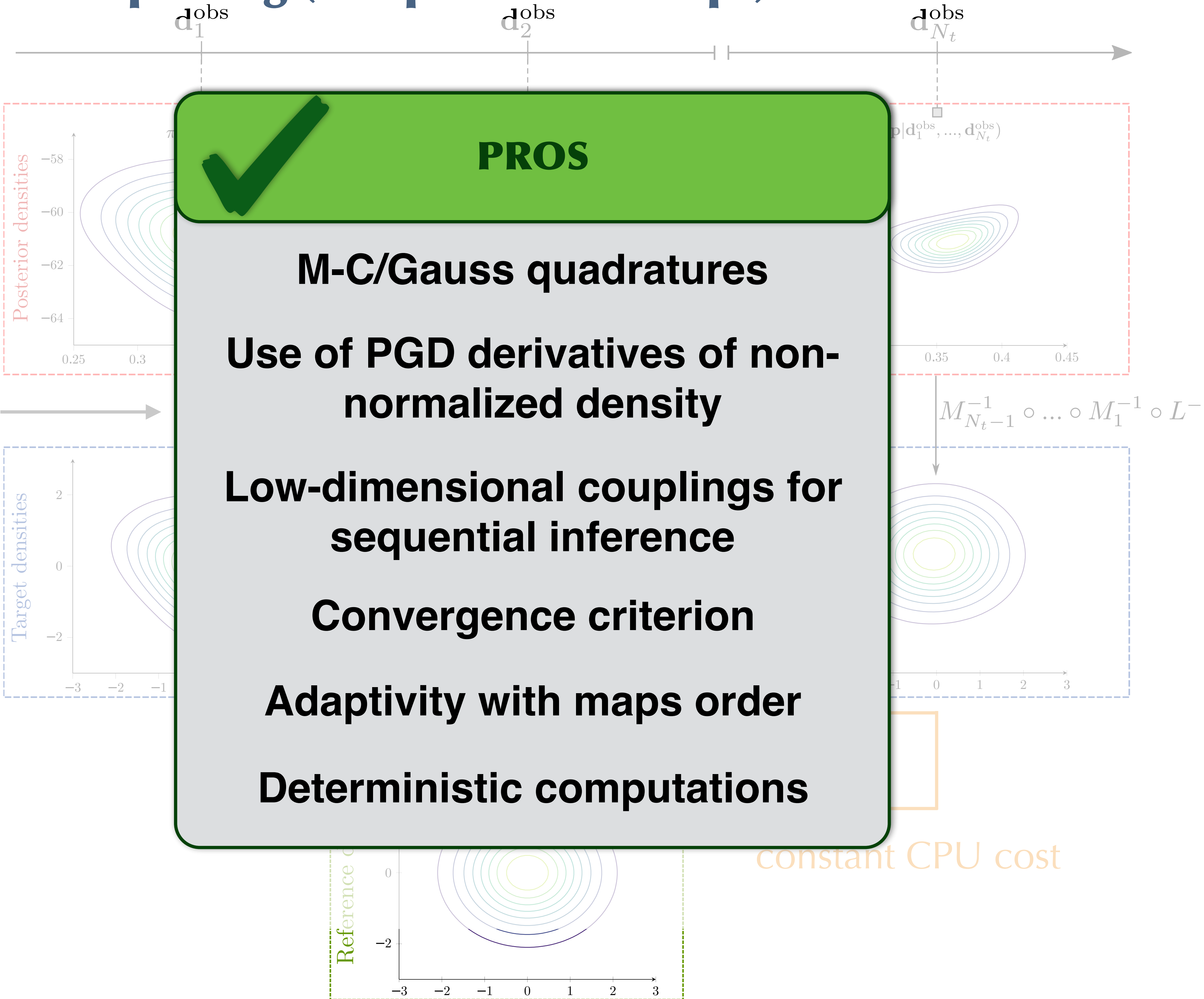
Variance diagnostic [Spantini *et al.* 18]

$$\epsilon_\sigma = \frac{1}{2}\mathbb{V}\text{ar}_\rho\left[\ln \frac{\nu_\rho}{M_\sharp^{-1}\nu_\pi}\right]$$

- sampling error estimate
- clear convergence criterion
- adaptive strategy on map order

Transport Map Sampling

Sequential updating (composition of maps)



Laplace approximation *(linear map for normalization)*

constant CPU cost

PROS

M-C/Gauss quadratures

Use of PGD derivatives of non-normalized density

Low-dimensional couplings for sequential inference

Convergence criterion

Adaptivity with maps order

Deterministic computations

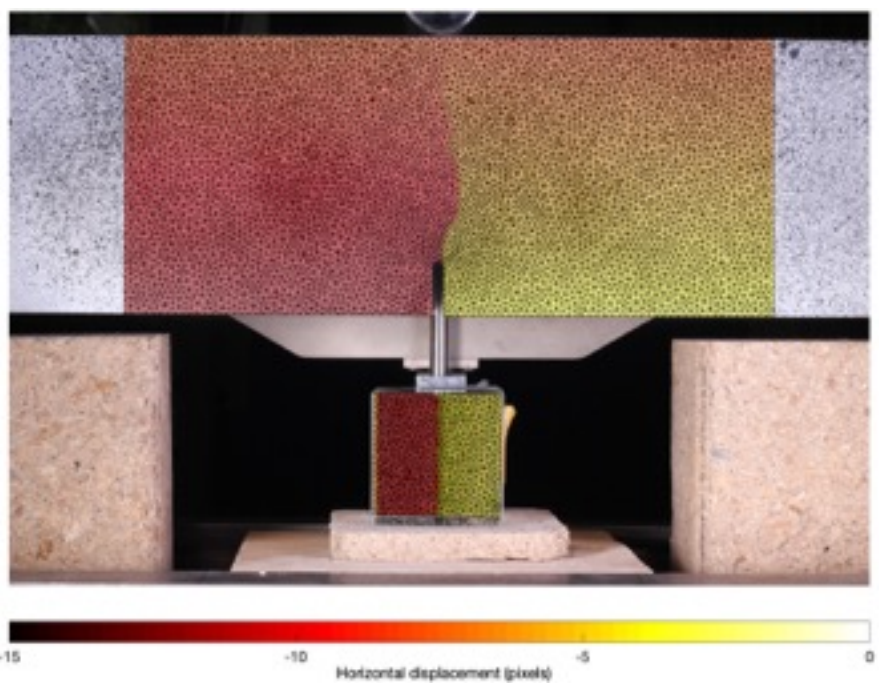
Results

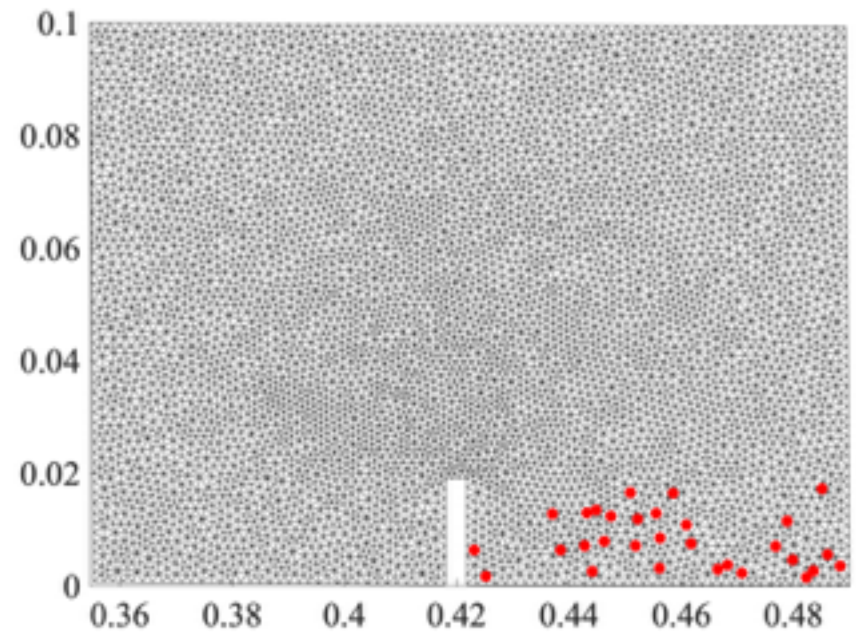
Assimilation with Transport Maps & PGD

$$\pi(\bar{Y}_0, \bar{A}_d|\mathbf{d}_1^{\text{obs}}, \ldots, \mathbf{d}_i^{\text{obs}}) \propto \prod_{j=1}^{i} \pi(\mathbf{d}_j^{\text{obs}}|\bar{Y}_0, \bar{A}_d).\pi_0(\bar{Y}_0, \bar{A}_d)$$

$$\epsilon_\sigma = 10^{-3}$$

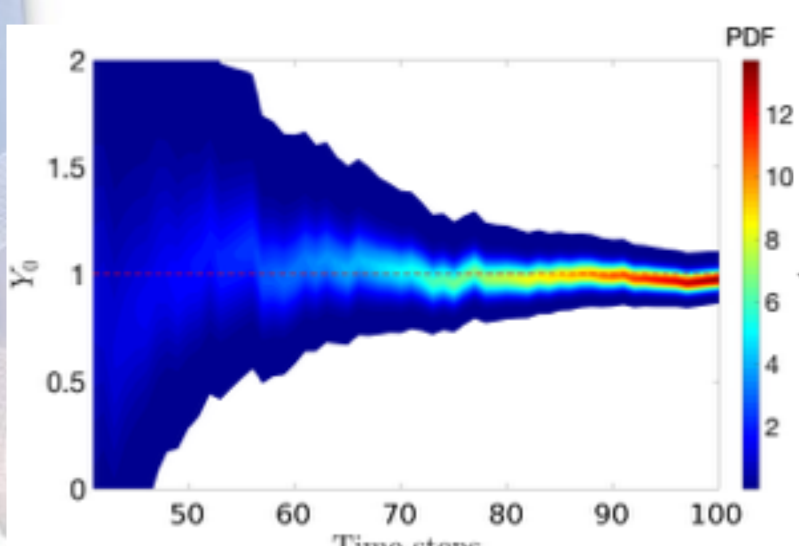
PGD approximation with $m = 6$

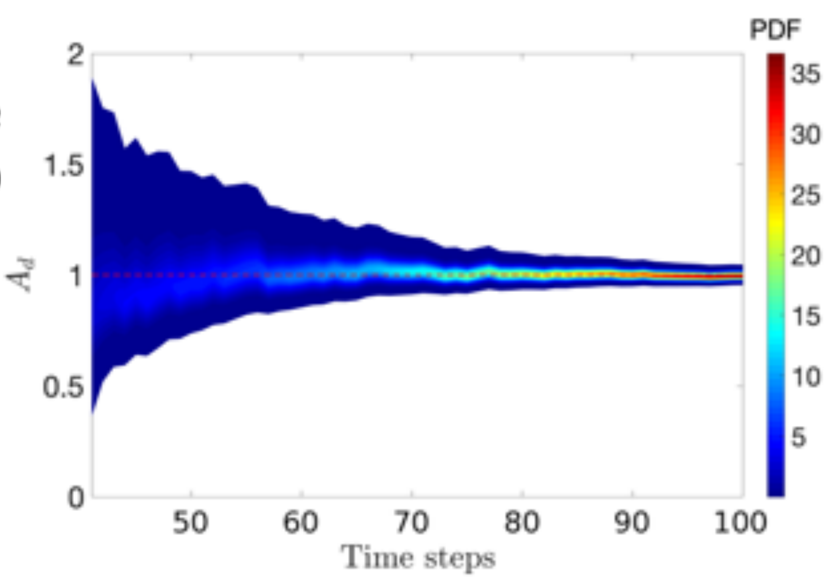


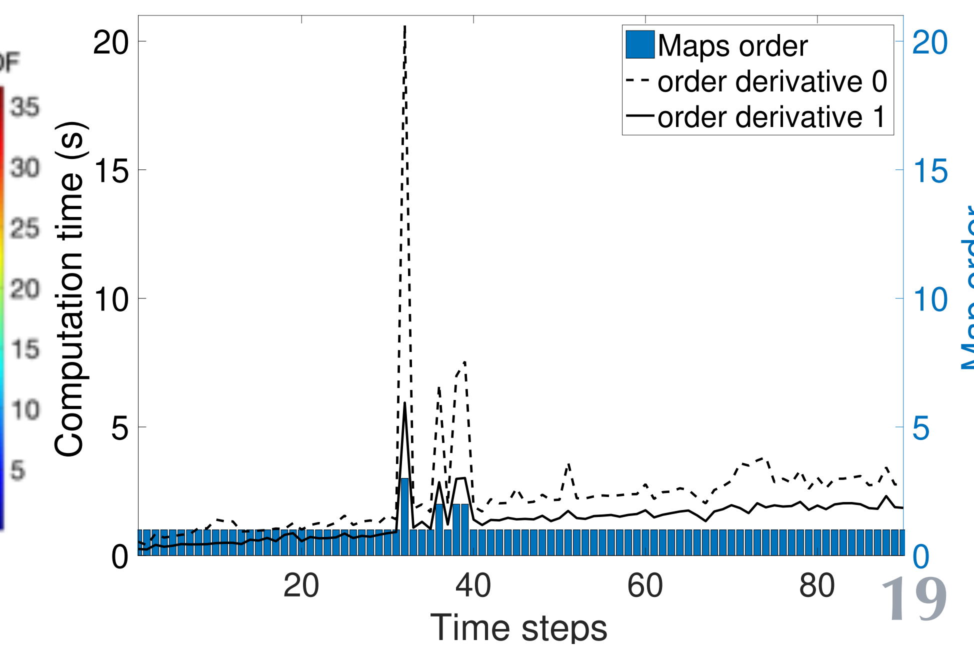


• Measurements points observed in the x direction

selection of most relevant DIC data (sensitivity analysis)



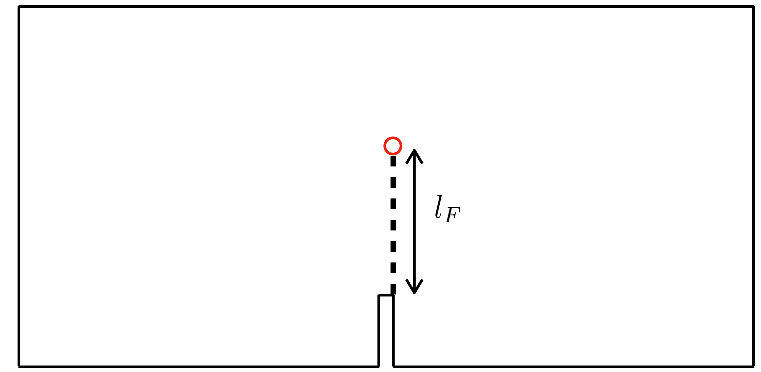


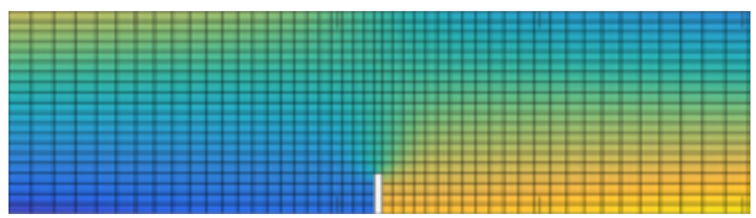


Post-processing

Kinematic bridge between damage & fracture mechanics

Goal: Predict the final crack propagation:

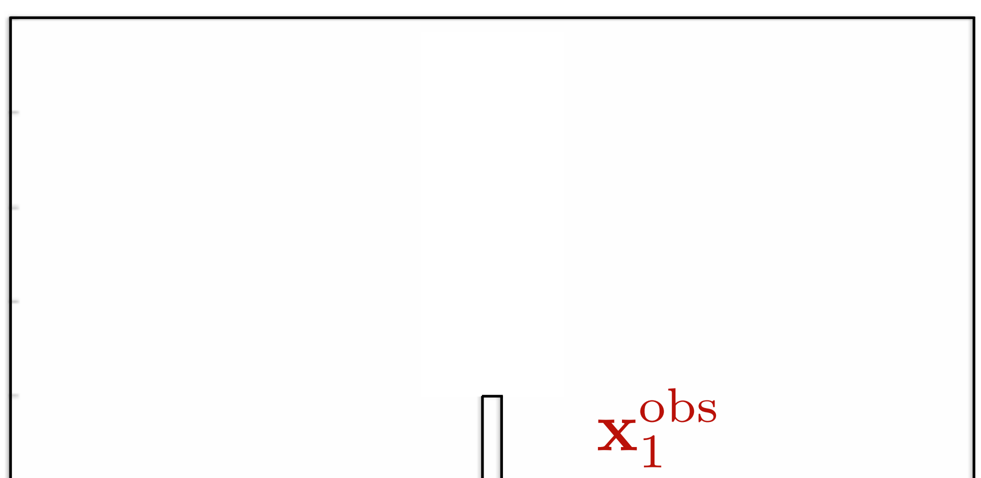


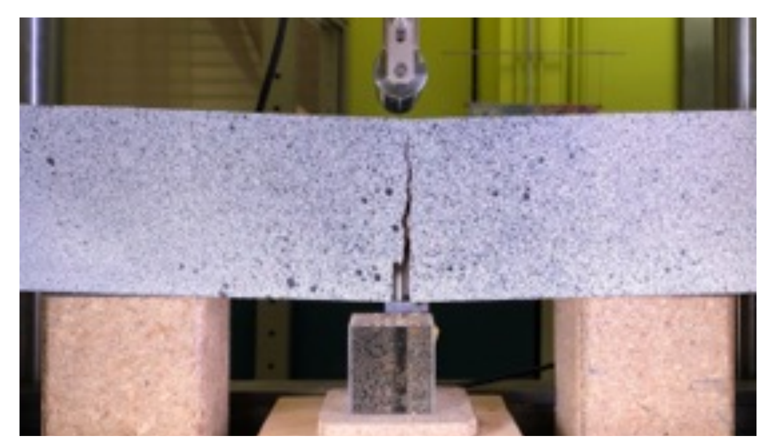


$\mathbf{u}_{\mathrm{PGD}}(\mathbf{x}, t_F, Y_0, A_d)$

$\mathbf{u}_{\mathrm{SVD}}(\mathbf{x}, l)$

$\mathbf{x} = \mathbf{x}_1^{\mathrm{obs}}$



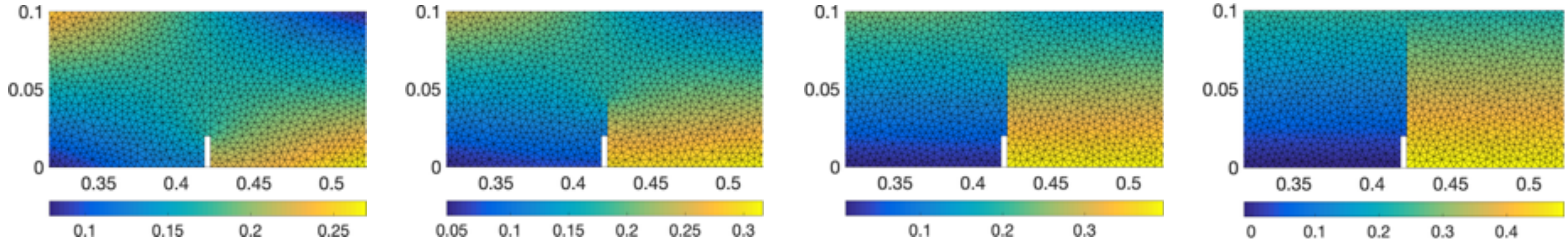


- Bayesian bridge: $\pi(l) = \pi_u(u^{\mathrm{SVD}}(l)).\pi_{\mathrm{pr}}(l)$

Results

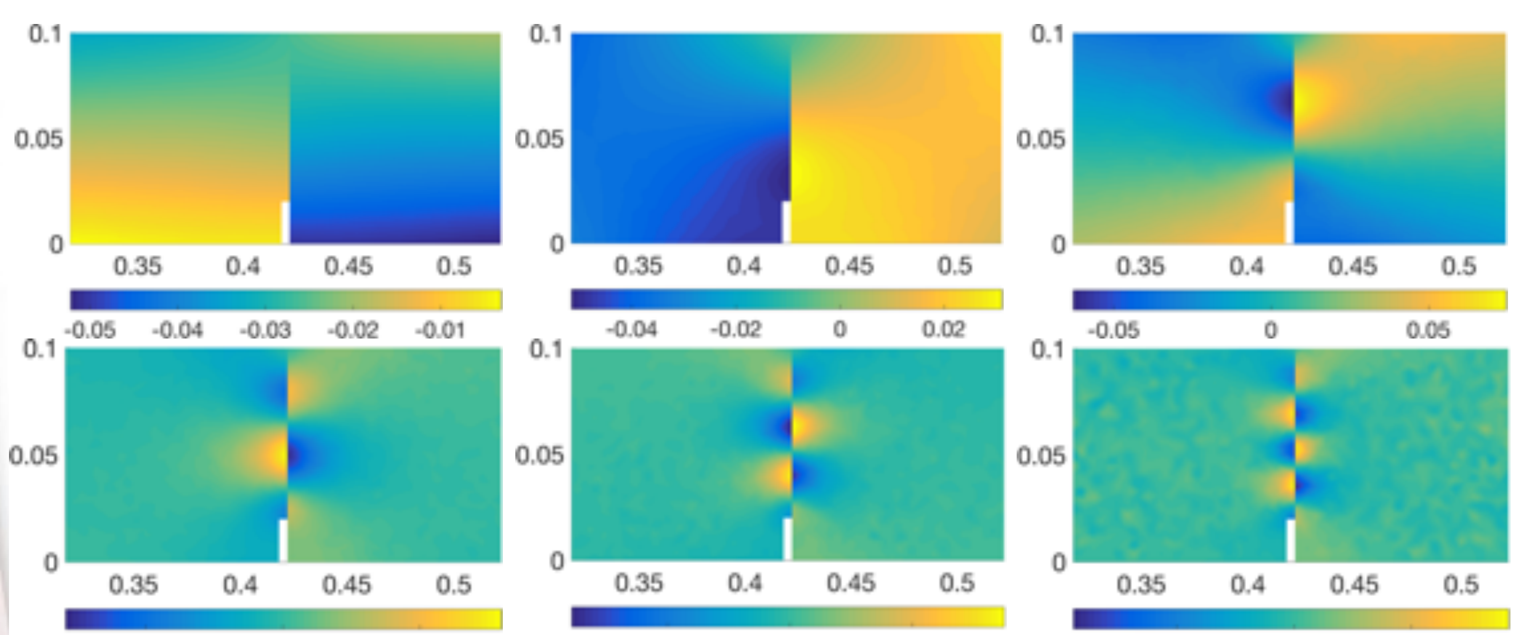
Kinematic bridge between damage and fracture mechanics

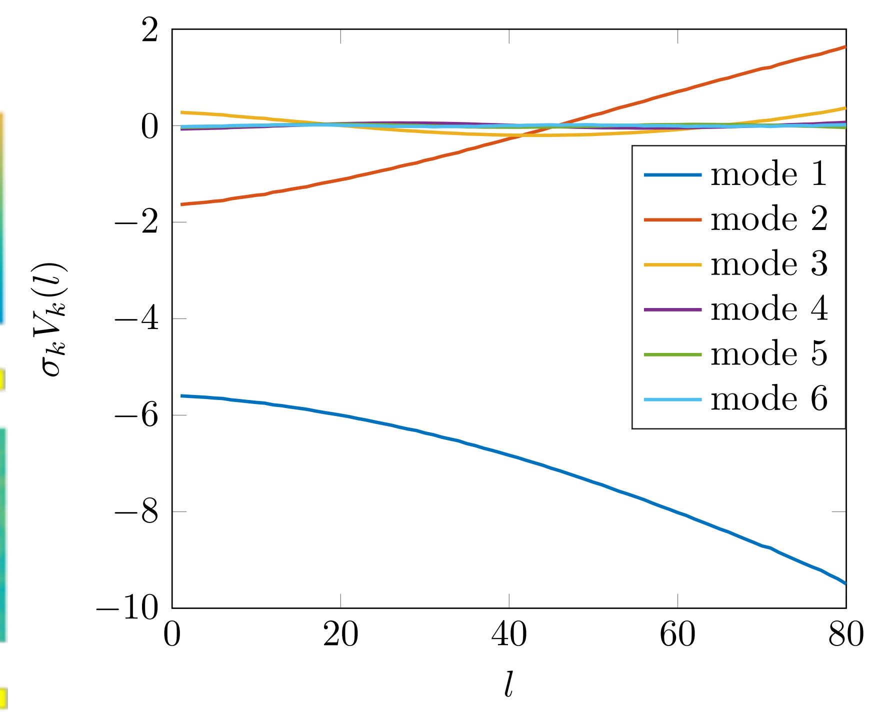
⟶ post-process of elastic solutions with varying crack length l (unit loading)



$$l \in \{0, 1, 2, \ldots, 79\} \longrightarrow \mathbb{Y} = \{\mathbf{Y}_1, \ldots, \mathbf{Y}_{80}\} \longrightarrow \mathbb{Y} = \mathbb{U}\mathbb{D}\mathbb{V}^T$$

$$\boxed{\mathbf{u}_{\mathrm{SVD}}(\mathbf{x}, l) = \sum_{k=1}^{N_{\mathrm{SVD}}} \sigma_k \mathbf{u}_k(\mathbf{x}) v_k(l)}$$ (meta-model constructed in the offline phase)

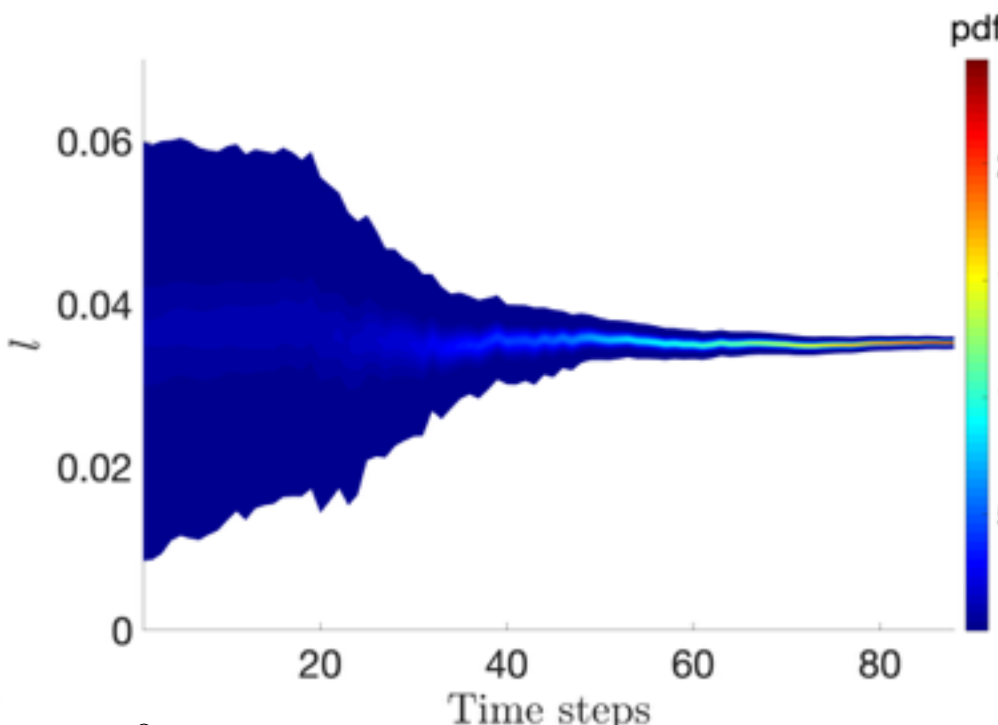


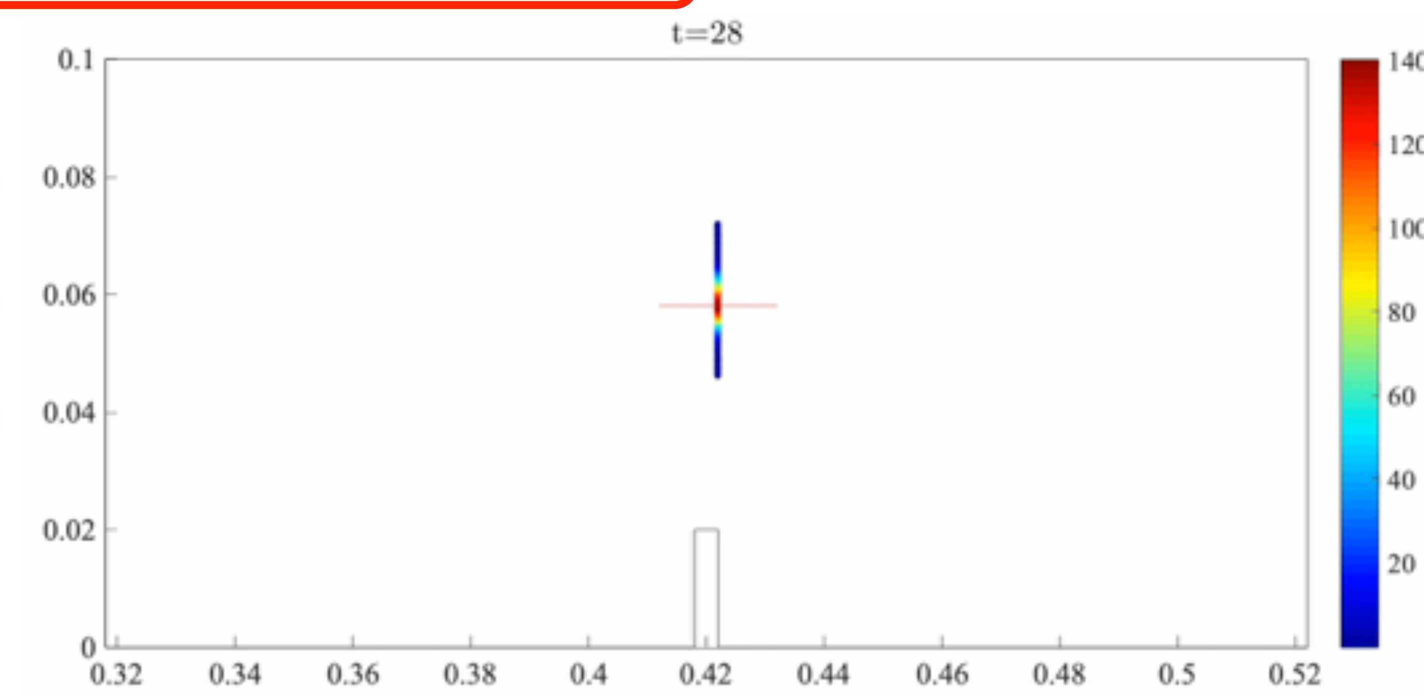


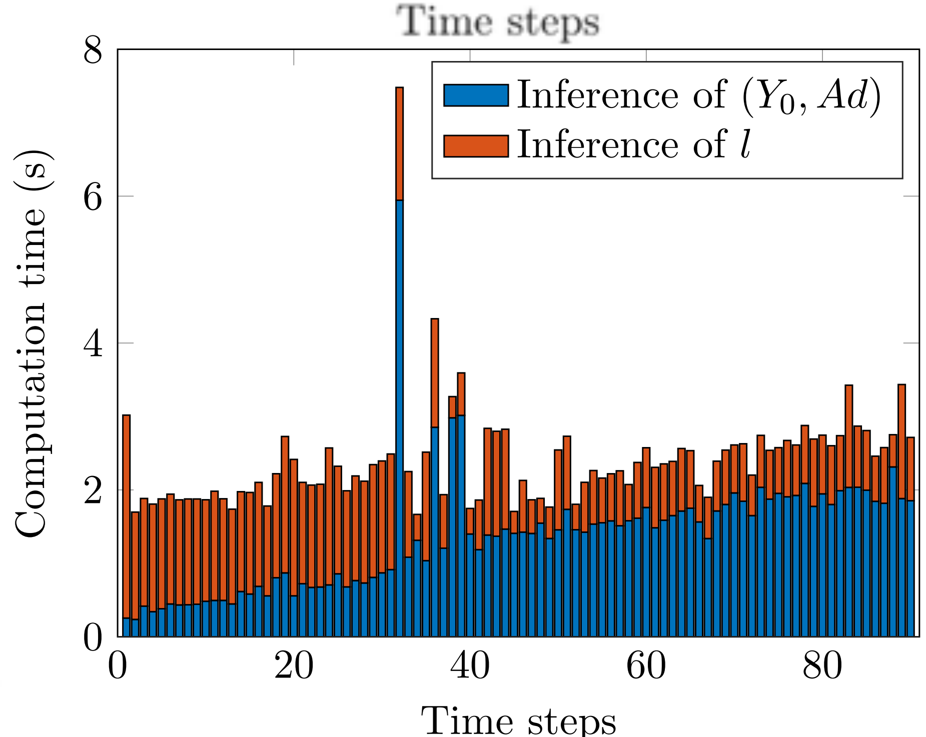
Results

On-the-fly prediction of the final crack length l_T

$$\boxed{\pi(l_T) = \pi_{\mathrm{u}}(\mathbf{u}^{\mathrm{SVD}}(l_T)).\pi_0(l_T)}$$







→ can be considered as real-time

Outline

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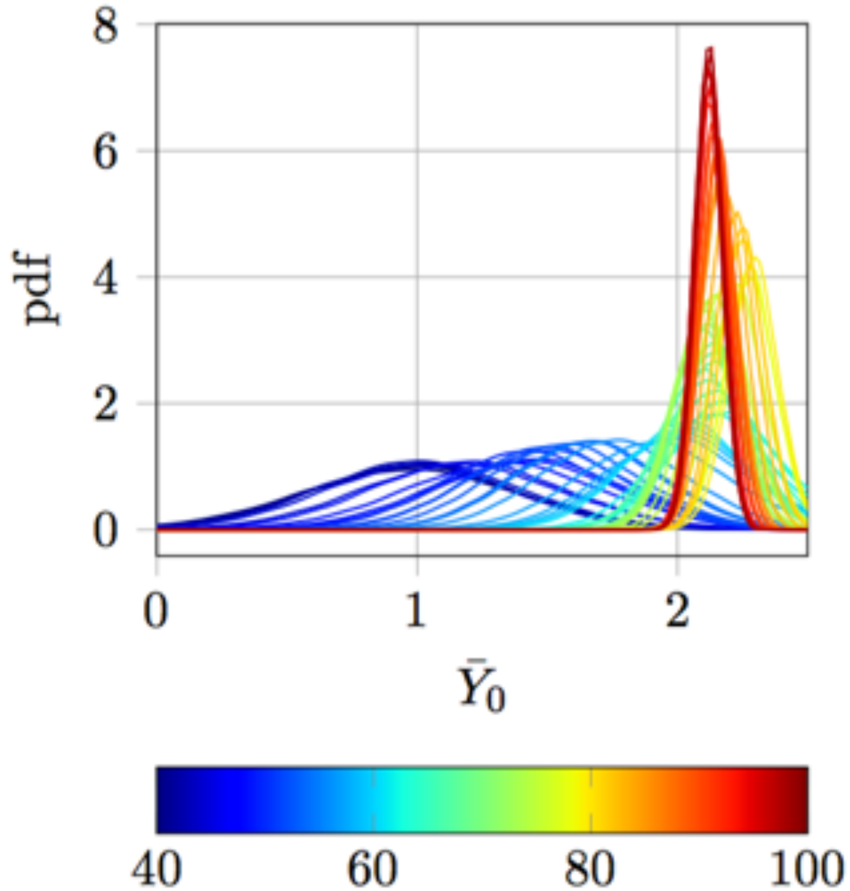
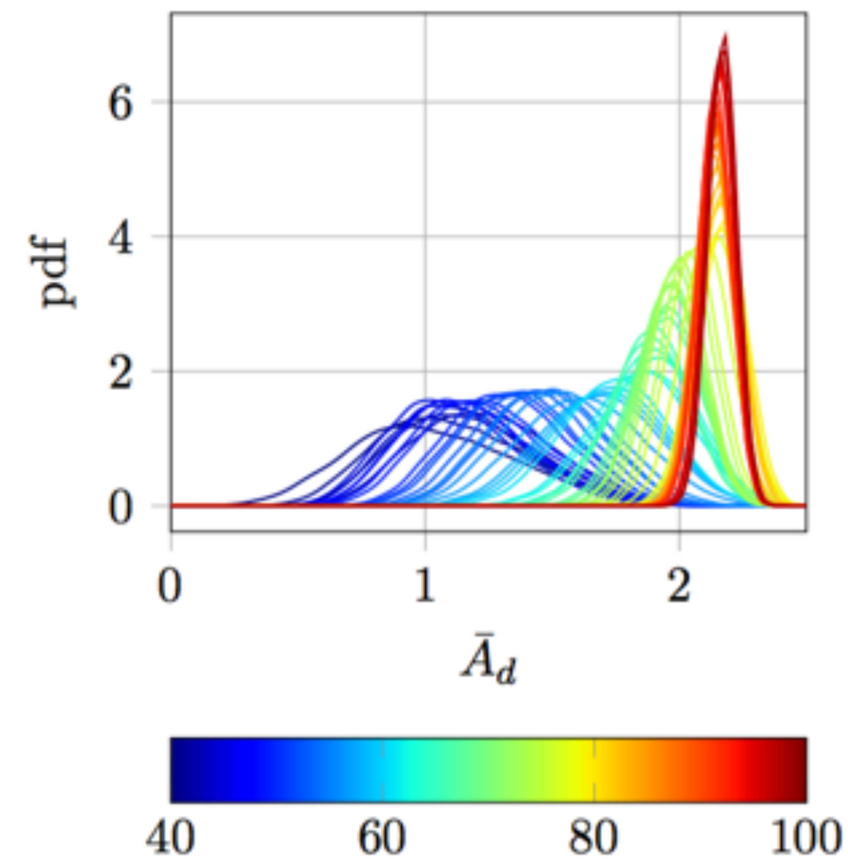
2. Real-time sampling using Transport Maps
3. Model bias correction from data
4. Feedback control

Bias Effects

- use of a surrogate PGD model with $m = 3$





⟶ influence limited during first time steps (elasticity with 1st mode alone)

⟶ divergence of the sequential data assimilation procedure (shifted marginals)

⟶ use of error estimates or high-fidelity models often not possible
[Calvetti *et al.* 18]

Bias Correction

→ **data-based enrichment**, comparing predicted outputs and actual data

→ defined dynamically and in a stochastic setting

extension of PBDW/hybrid twins [Maday *et al.* 15, Chinesta *et al.* 18]

Stochastic residual (computable)

$$\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_i) = \mathbf{d}_i^{\mathrm{obs}} - \mathbf{e}_{\mathrm{meas}} - \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, t_i, \mathbf{p})$$

spatial coordinates of measurement points

Corrected model

$$\mathcal{M}^{\mathrm{corr}}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}) = \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}) + \hat{\mathbf{B}}_{i\to i+1}(\mathbf{x}^{\mathrm{obs}})$$

extrapolated model bias
(Gaussian pdf)

$$\boxed{\pi(\mathbf{d}_{i+1}^{\mathrm{obs}}|\mathbf{p}) = \pi_{\hat{B}}(\mathbf{d}_{i+1}^{\mathrm{obs}} - \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}))}$$

Bias Correction

Extrapolation procedure

- linear independent extrapolation of mean and standard deviation of $\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_i)$

⟶ no physics consideration / inconsistent results (noise extrapolation)

- global linear extrapolation involving physics (filtering noise)

$$\mathbb{B}_{\mathrm{mean}} = \left[\mathrm{mean}\left(\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_1)\right), \ldots, \mathrm{mean}\left(\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_i)\right)\right] \qquad \mathbb{B}_{\mathrm{std}} = \left[\mathrm{std}\left(\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_1)\right), \ldots, \mathrm{std}\left(\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_i)\right)\right]$$

SVD decomposition (+truncation)

$$\mathbb{B}_{\mathrm{mean}} = \mathbb{U}_{\mathrm{mean}}\mathbb{D}_{\mathrm{mean}}\mathbb{V}^T_{\mathrm{mean}} \qquad \mathbb{B}_{\mathrm{std}} = \mathbb{U}_{\mathrm{std}}\mathbb{D}_{\mathrm{std}}\mathbb{V}^T_{\mathrm{std}}$$

⟶ linear extrapolations of time SVD modes: $\hat{\mathbf{V}}_{\mathrm{mean}}$ $\hat{\mathbf{V}}_{\mathrm{std}}$

⟶ *use of the Sequential-Karhunen-Loeve (SKL) method* [Ross *et al.* 08]

fast SVD decomposition of $[\mathbb{M}_{[1,i-1]} \ \mathbb{M}_i]$ *knowing that of* $\mathbb{M}_{[1,i-1]}$

⟶ recovery of $\hat{\mathbf{B}}_{i \to i+1}(\mathbf{x}^{\mathrm{obs}})$ with truncated SVD

Indication on Model Bias

$$\boxed{\pi(\mathbf{p}|\mathbf{d}^{\mathrm{obs}}) = \frac{1}{C}\pi(\mathbf{d}^{\mathrm{obs}}|\mathbf{p}).\pi_0(\mathbf{p})}$$

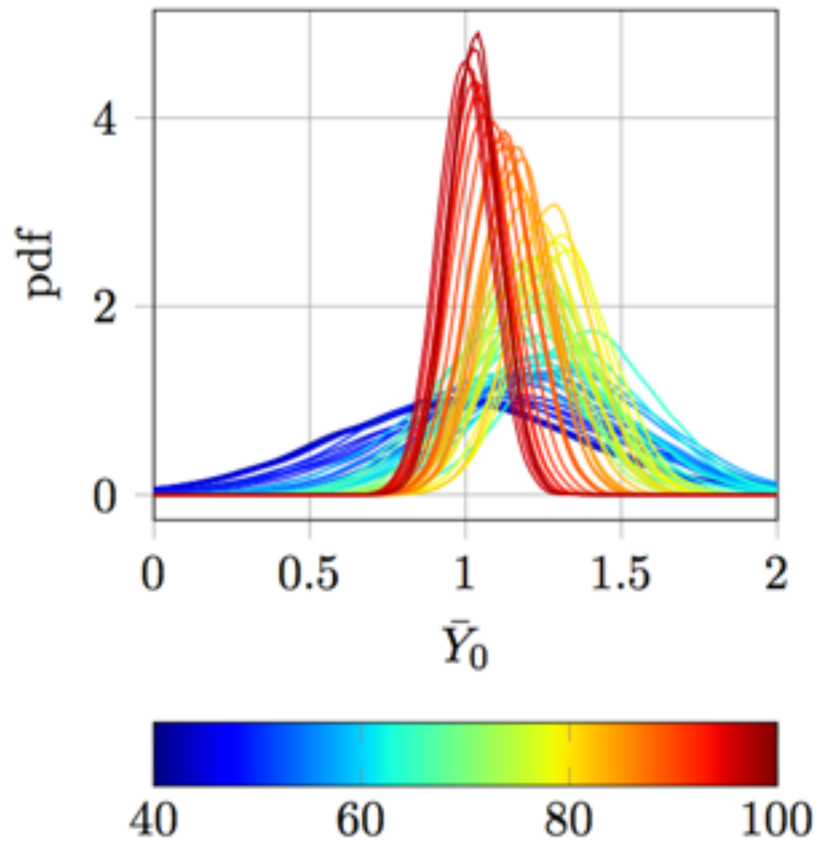
$$C = \int \pi(\mathbf{d}^{\mathrm{obs}}|\mathbf{p}).\pi_0(\mathbf{p})\mathrm{d}\mathbf{p} = \pi(\mathbf{d}^{\mathrm{obs}})$$ **: model evidence**

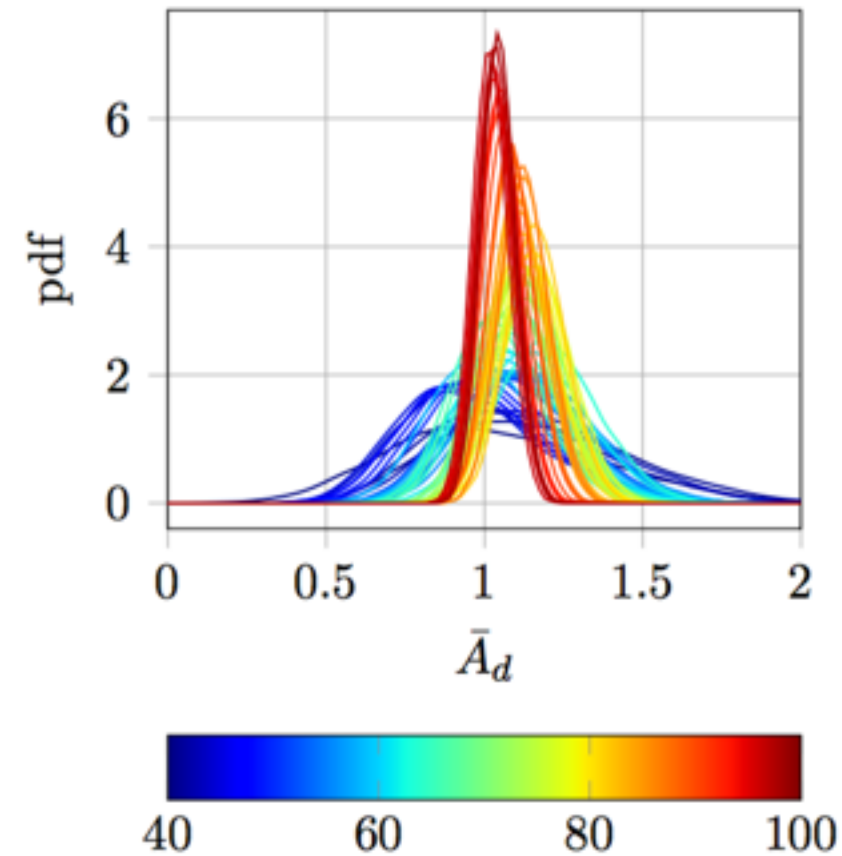
$$= \exp\left(\mathbb{E}_\rho\left[\log\left(M_\sharp^{-1}\pi\right) - \log(\rho)\right]\right)$$ [El Moselhy & Marzouk 12]

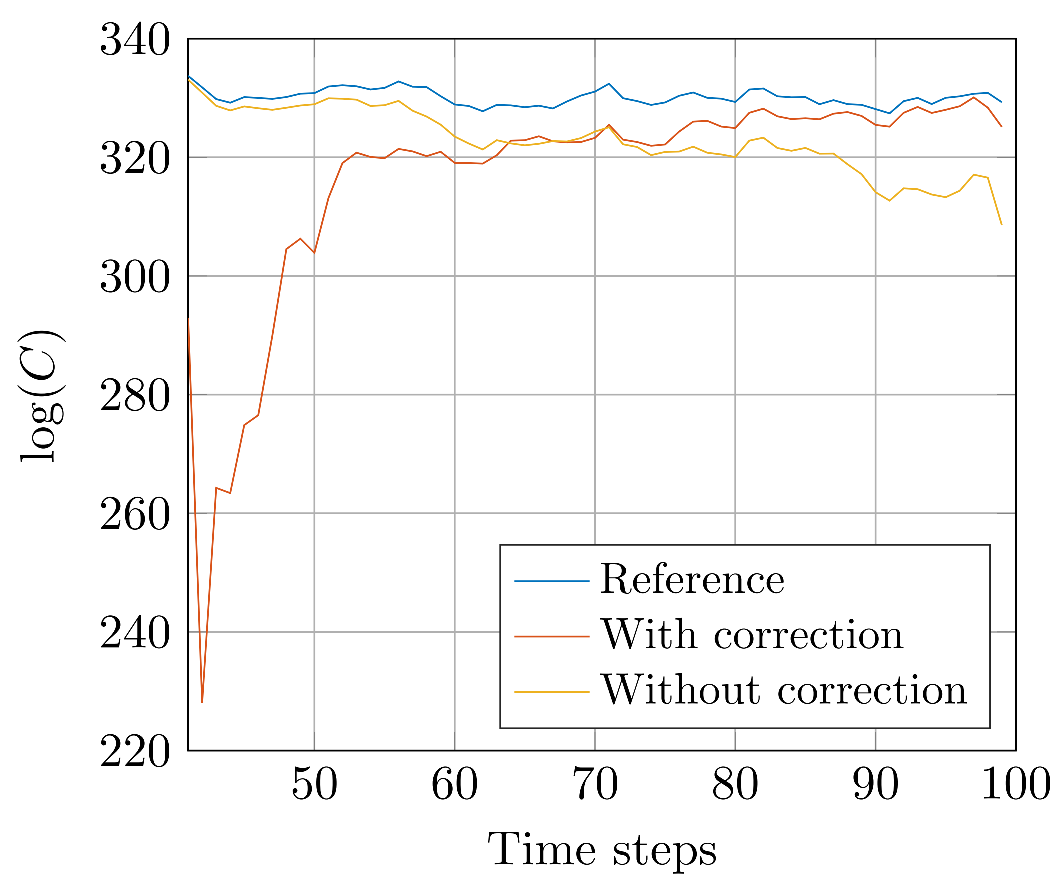
→ decreases when the model becomes inaccurate

- → evolution of C monitored along the assimilation process
- → implementation of the correction **when C drops drastically**
- → if model inaccurate in a given time range, corresponding maps removed

Results


pdf
4
2
0
0
0.5
1
1.5
2
$\bar{Y}_0$
40
60
80
100


pdf
6
4
2
0
0
0.5
1
1.5
2
$\bar{A}_d$
40
60
80
100


340
320
300
280
260
240
220
log(C)
50
60
70
80
90
100
Time steps
Reference
With correction
Without correction

Outline

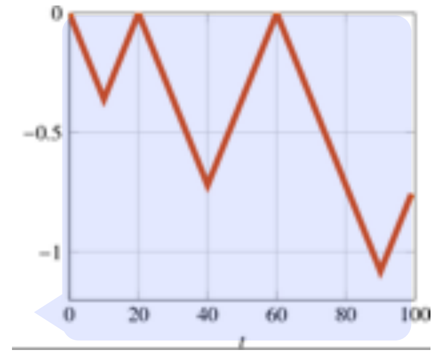
1. Bayesian formulation with ROM

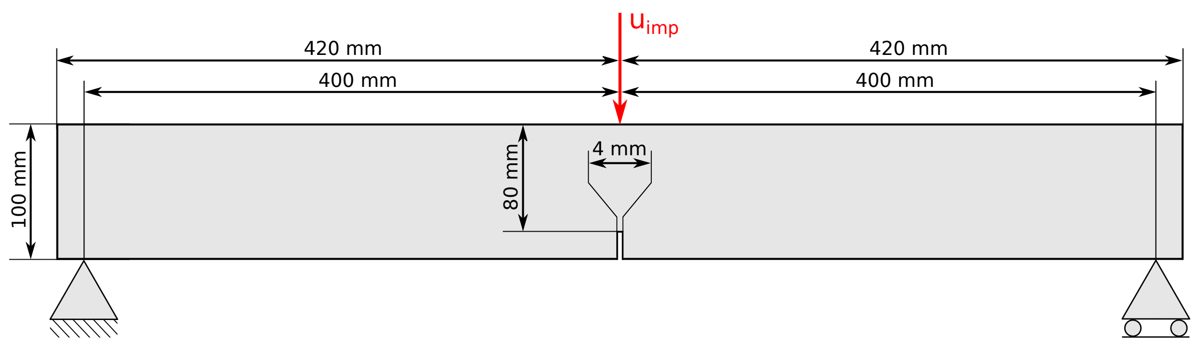
2. Real-time sampling using Transport Maps

3. Model bias correction from data

4. Feedback control

Control example

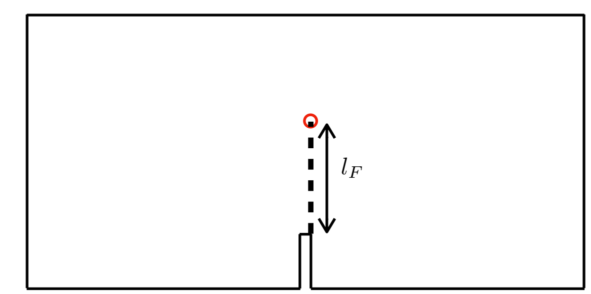




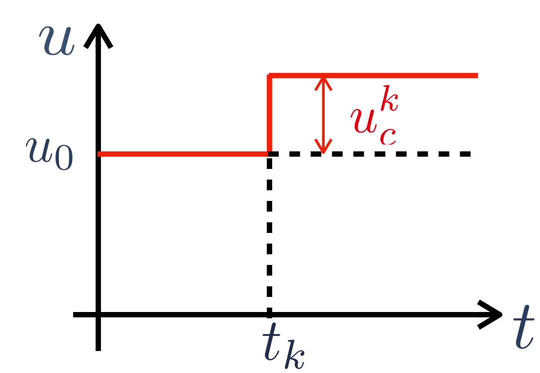
Goal: Control the final length of the crack (at the end of the loading history)

- Quantity of interest:

$$q = mean[l_F(T, \mathbf{p}, u)] - 3.std[l_F(T, \mathbf{p}, u)]$$



- Objective: $q_{obj} = 1$ (probability to be after prescribed position = 0.99)



- Control variable: magnitude of each loading cycle

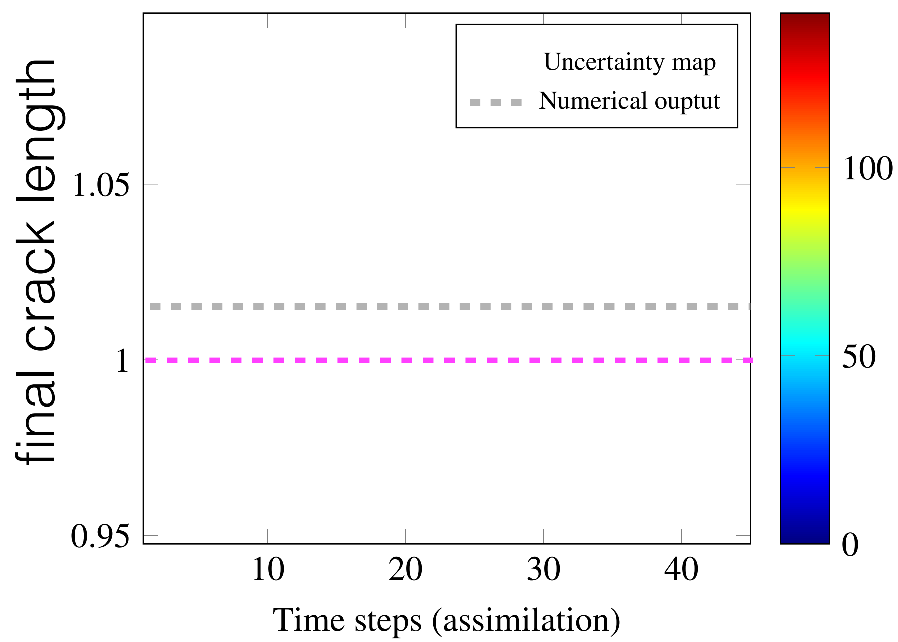
- Computation of the control variable ⟶ simple

propagating uncertainty on parameters through the PGD model & TM sampling

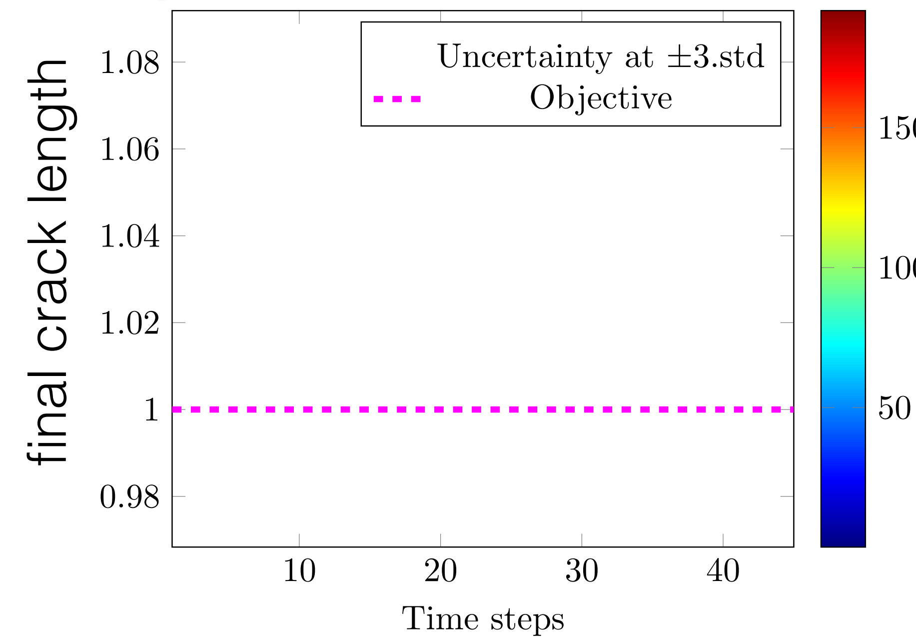
Control example

Results

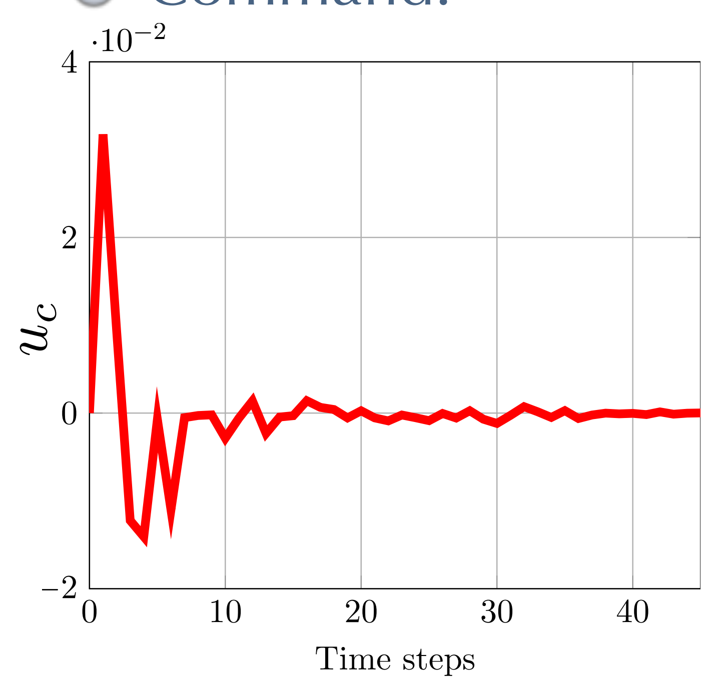
- Without control:



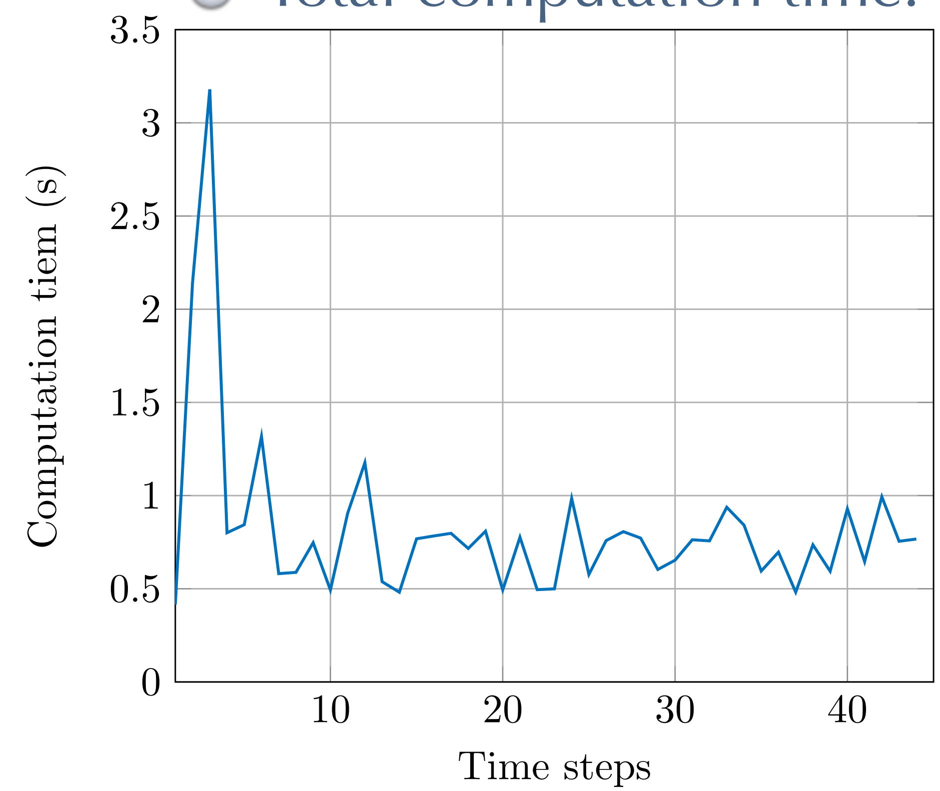
- With control:



- Command:



- Total computation time:



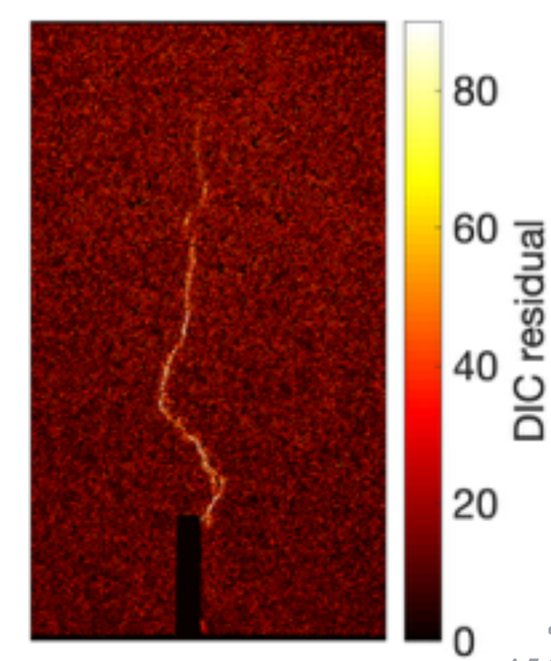
Conclusions & Prospects

Conclusions:

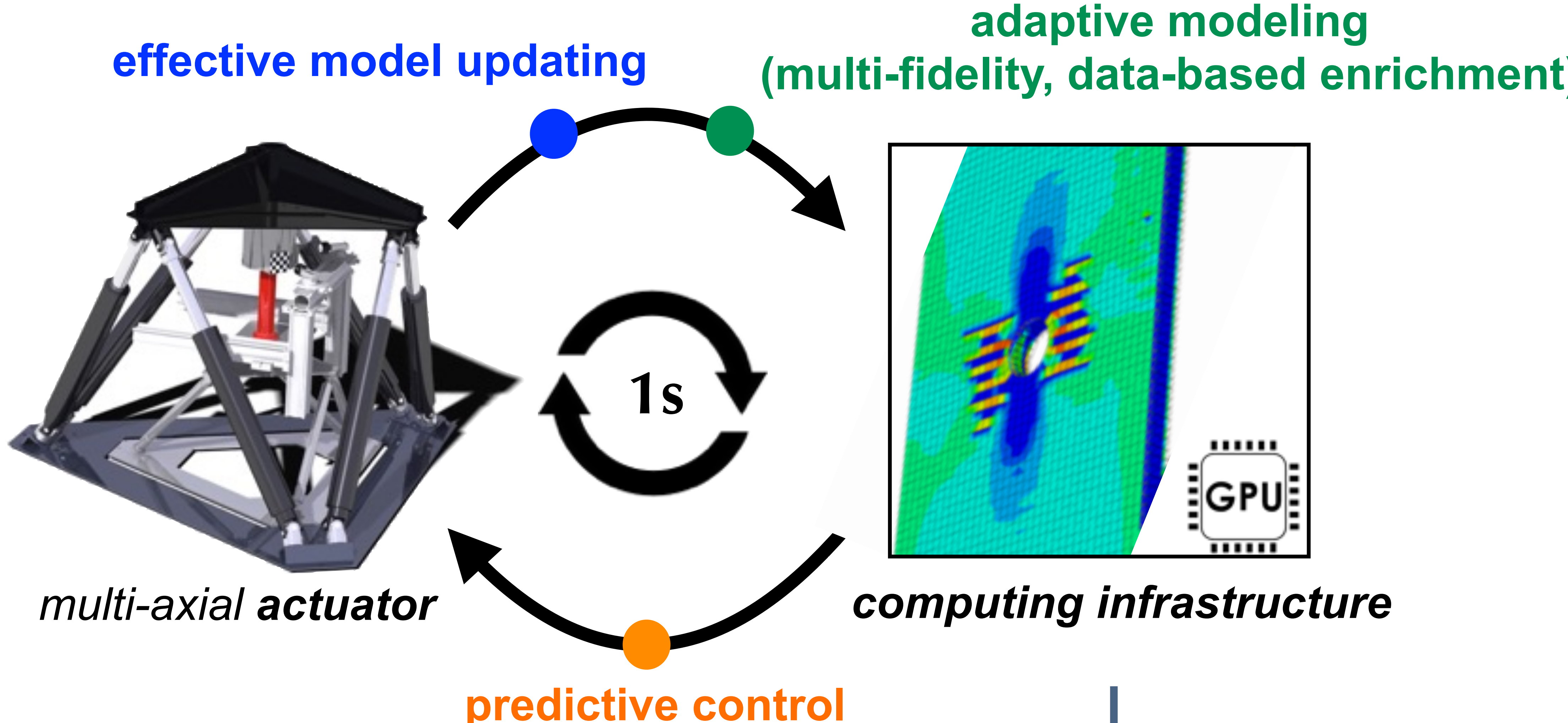
- Transport Maps allow deterministic computations to characterize posteriors
- A constant CPU cost is involved along the sequential updating
- PGD allows to get fast computations
- Consistent procedure for modeling error (bias) correction
- Application on a nonlinear damage problem (single crack)

Prospects:

- Improve automatic order adaptivity of maps in TM sampling
- Regression of the maps composition
- Improve the description of crack propagation
- More complex constitutive models (e.g. composites)
- Investigate high-dimension problems (field identification)
- System control with UQ (damage)
- Interpret model enrichment (learning/mining)



New Project



↓

real-time,
robust, portable

➡ SMART ENGINEERING STRUCTURES

Thank you!!

- P-B. Rubio, F. Louf, L. C., Fast model updating coupling Bayesian inference and PGD model reduction, *Computational Mechanics, 62(6):1485-1509* (2018)
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- P-B. Rubio, L. C., F. Louf, Real-time data assimilation and control on mechanical systems under uncertainties, *Advanced Modeling and Simulation in Engineering Sciences, accepted* (2021)