PERSISTENT HOMOTOPY GROUPS OF METRIC SPACES

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Abstract. We study notions of persistent homotopy groups of compact metric spaces together with their stability properties in the Gromov-Hausdorff sense. We pay particular attention to the case of fundamental groups for which we obtain a more precise description.

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1. Introduction

1.1. **Motivation.** In [Gro99], Gromov motivated the study of quantitative homotopy theory with a story, where he got different responses from two topologists about the same question whether the universe is simply-connected. This discussion inspired Gromov to think, instead of asking a 'yes or no' question, it can be more interesting to ask HOW simply-connected a space is. In Figure 1, the space X is given by a 2-sphere attached with a tiny handle, which is clearly not simply-connected. However, noticing how small the handle is compared with the whole space X, one may think that X is 'almost' simply-connected in a reasonable sense. In this paper, we provide a way to measure simply-connectedness in a similar spirit.

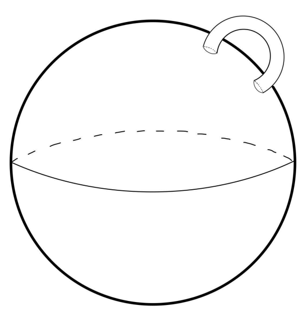

Figure 1. The space X given by a 2-sphere attached with a tiny handle.

1.2. **Related work.** In this section we overview related work.

Quantitative topology and persistence. The idea of *topological persistence* has independently appeared in different settings. In a nutshell, via some functorial constructions one assigns to a topological space X: (1) for every $\epsilon > 0$ some space $S_\epsilon(X)$ and (2) for every $\alpha \geqslant \epsilon$ an equivalence relation $R_\alpha(X)$ on $S_\epsilon(X)$. For example, when X is a metric space, then $S_\epsilon(X)$ could be the collection of k-cycles of its Vietoris-Rips complex $\mathrm{VR}_\epsilon(X)$ and $R_\alpha(X)$ the equivalence relation arising from two cycles $(c, c') \in R_\alpha(X)$ whenever there exists a $(k+1)$-chain in $R_\alpha(X)$ such that its boundary is the sum $c + c'$.

Ideas of this type can already be found in the work of Borsuk in the 1940s. For instance, in [Bor55, Section 3] Borsuk studies the transfer of certain scale dependent topological properties of compacta under a suitable metric.

Frosini and collaborators [Fro90, Fro92] and Robins [Rob99] identified ideas related to concepts nowadays known under the term *persistence* in the the context of applications. From an algorithmic perspective, the study of topological persistence was initiated by Edelsburnner et al. in [ELZ00]. See the overviews [EH08, Car09, Ghr08, Wei11], for more information.

Scale dependent invariants which blend topology with geometry have also been considered by Gromov [Gro99, Gro07]. For example, in the late 1990s Gromov suggested the study of homotopy invariants in a scale dependent manner. One question posed by Gromov was: given a λ-Lipschitz and contractible map $f : X \to Y$ between metric spaces X and Y, determine the function $\lambda \mapsto \Lambda(\lambda)$ such that there exists a $\Lambda(\lambda)$ null-homotopy of f. This question can be interpreted as a homotopical version of the homological question posed at the beginning of this section.

Homotopy groups and persistence. In the late 1990s Frosini and Mulazzani [FM99] considered a construction which blended ideas related to persistence with homotopy groups. Their construction dealt with pairs (M, φ) where M is a closed manifold and $\varphi = (\varphi_1, \ldots, \varphi_k) : M \to \mathbb{R}^k$ is a continuous function. In this setting, for $\zeta, \eta \in \mathbb{R}^k$ with $\zeta_i \leqslant \eta_i$ for all i they considered a certain $(\varphi\zeta, \eta)$-parametrized version of the fundamental group arising from considering equivalence classes of based loops in sublevel sets $M_\zeta := \{x \in M | \varphi_i(x) \leqslant \zeta_i, \forall i\}$ under the equivalence relation: for α and β pointed loops in M_ζ, $\alpha \sim_\eta \beta$ iff there is a (pointed) homotopy whose tracks remain in M_η. These groups are then used in order to obtain lower bounds for the so called natural pseudo distance between two manifolds. A more comprehensive look at several related ideas is carried out by Frosini in [Fro99].

In [Let12] in his study of knots, Letscher considered the scenario in which one is given a filtration $\{X_\epsilon\}_{\epsilon>0}$ of a path connected topological space X and for each $\epsilon, \delta > 0$ he considered the group $\pi_1^{(\delta)}(X_\epsilon)$ consisting of equivalence classes of (based) loops contained in X_ϵ under the equivalence relation stating that two loops in X_ϵ are equivalent iff there is a homotopy between them which is fully contained in $X_{\epsilon+\delta}$.

In [BCW14] Barceló et al. introduced a certain scale dependent notion of discrete homotopy groups associated to metric spaces via the study of 1-Lipschitz maps from n-cubes into the metric space in question. Barceló et. al explicitly studied the behaviour of their invariants under scale coarsening.

In his dissertation [Wil11], Wilkins studies a certain notion of discrete fundamental group which he uses to in turn induce a notion of critical values of a metric space. This notion differs from the one introduced in [BCW14] in that the respective notions of homotopies between discrete loops are different.

The notion studied by Wilkins originated in work by Berestovskii and Plaut [BP01] in the context of topological groups and subsequently extended in [BP07] to the setting of uniform spaces. In [BP07] the authors also consider discrete fundamental groups as an inverse system and study its inverse limit. See [Pla07, Page 599, Plaut] for a discussion about the relationship between Vietoris-Rips complex and discrete fundamental groups.

In [BL17, Section 8.1] Blumberg and Lesnick define persistent homotopy groups of $\mathbb{R}$-spaces (in a way somewhat different from Letscher's) and formulate certain persistent Whitehead conjectures. In [BPV19] Batan, Pamuk and Varli study a certain persistent version of Van-Kampen's theorem for Letscher's definition of persistent homotopy groups of $\mathbb{R}$-spaces.

In [Rie17] Rieser studies a certain version of discrete homotopy groups of Čech closure spaces. A Čech closure space is a set equipped with a Čech closure operator, which is a concept similar to that of a topological closure operator except that it is not required to be idempotent. Given a finite metric space X, a way of obtaining Čech closure operators is the following: for each $r \geqslant 0$ let c_r be a map between subsets of X such that $c_r(A) = \{x \in X : d(x, A) \leqslant r\}$ for each $A \subset X$. The author shows that each c_r is a Čech closure operator and the identity map $(X, c_q) \to (X, c_r)$ induces maps $\iota_n^{q,r} : \pi_n(X, c_q) \to \pi_n(X, c_r)$ for every $q \leqslant r$ and $n \geqslant 0$. Then he defines the persistent homotopy groups of X to be the collection $\{\mathrm{Im}(\iota_n^{q,r})\}_{r \geqslant q \geqslant 0}$.

In [Jar19] Jardine utilizes poset theoretic ideas to study Vietoris-Rips complexes (and some variants) associated to data sets and provides explicit conditions for the poset morphism induced by an inclusion of data sets to be quantifiably close to a homotopy equivalence. Jardine also considers persistent fundamental groupoids and higher-dimensional persistent homotopy in the lecture notes [Jar20].

1.3. **Overview of our results.** We consider three different definitions of persistent fundamental groups. One of them, denoted $\mathrm{P\Pi}_1^{\mathrm{K},\bullet}$ arises from isometrically embedding a given metric space X, via the so called Kuratowski embedding, into $L^\infty(X)$ and then applying the π_1 functor to successive *thickenings* X^ϵ of X inside $L^\infty(X)$, cf. Definition 5.5. This definition is similar in spirit to the construction used by Gromov for defining the so-called filling radius invariant [G+83]. Another definition, denoted $\mathrm{P\Pi}_1^{\mathrm{VR},\bullet}$ we consider stems from simply considering the geometric realization of nested Vietoris-Rips simplicial complexes for different choices of the scale parameter and then applying the π_1 functor, cf. Definition 5.7. The third, more combinatorial, definition (cf. Definition 5.2) which we consider is one coming from the work of Plaut and Wilkins [PW13] (see also Barcelo et al [BCW14]). This construction, denoted simply $\mathrm{P\Pi}_1^\bullet$, turns out to be isomorphic to the edge path groups of the nested Vietoris-Rips simplicial complexes used in the previous definition (see Theorem 5.12).

All these three definitions can be constructed in two versions: the open version and the close version, which will be specified with the supscripts $<$ and $\leqslant$ respectively. Throughout this paper, we mainly work on the closed version and will omit the supscripts unless necessary. Now we observe that these three definitions give rise to isomorphic persistent fundamental groups.

Theorem 5.12 (Isomorphisms of persistent fundamental groups)**.** *Given a pointed compact metric space (X, x_0), we have*

$$\mathrm{P\Pi}_1^{\mathrm{K},\bullet}(X, x_0) \cong \mathrm{P\Pi}_1^{\mathrm{VR},2\bullet}(X, x_0) \cong \mathrm{P\Pi}_1^{2\bullet}(X, x_0), \tag{2}$$

where the second isomorphism is true in either version, and the first isomorphism is true for the open version and for the closed version with X a finite metric space. In either version, $\mathrm{P\Pi}_1^{\mathrm{K},\bullet}(X, x_0)$ and $\mathrm{P\Pi}_1^{\mathrm{VR},2\bullet}(X, x_0)$ have homotopy-interleaving distance zero.

For a given compact metric space X and a base point $x_0 \in X$ denote

$$\mathrm{P\Pi}_1(X, x_0) = \left\{ \pi_1^\epsilon(X, x_0) \xrightarrow{\Phi_{\epsilon,\epsilon'}} \pi_1^{\epsilon'}(X, x_0) \right\}_{\epsilon' \geqslant \epsilon > 0}.$$

It is not difficult to see that under fairly mild assumptions, the structure maps $\Phi_{\epsilon,\epsilon'}$ of persistent fundamental groups of a geodesic space X are surjective and that for sufficiently small scale parameters the groups involved in the persistent fundamental group of X are isomorphic to its fundamental group $\pi_1(X)$. These two facts when combined suggest that as the scale parameter increases homotopy classes in X can only become equal (no new classes can appear), thus suggesting an existence of a tree-like structure (i.e. a dendrogram) underlying $\mathrm{P}\Pi_1(X)$. For instance, when X is $\mathbb{S}^1$, the geodesic unit circle, associated to $\mathrm{P}\Pi_1(\mathbb{S}^1)$ we have a dendrogram over $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ shown in Figure 2. The general situation is stated in the theorem below.

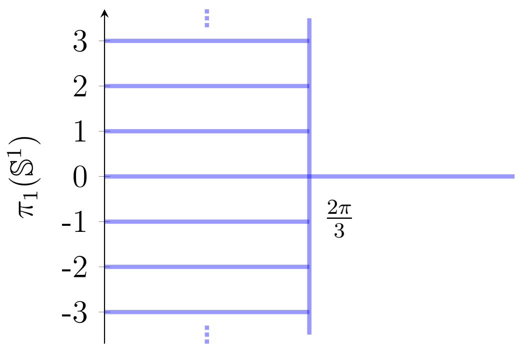

FIGURE 2. The dendrogram associated to $\mathrm{P}\Pi_1(\mathbb{S}^1)$. The y-axis represents elements of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, i.e., homotopy classes of continuous loops in $\mathbb{S}^1$ (each corresponds to an integer).

Theorem 5.15 (Dendrogram over $\pi_1(X)$)**.** *Let a compact geodesic metric space X be semi-locally simply connected (s.l.s.c.). Then there is a dendrogram $\theta_{\pi_1(X)}$ over $\pi_1(X)$, given by*

$$\theta_{\pi_1(X)}(\epsilon) := \begin{cases} \pi_1^{\epsilon}(X), & \text{if } \epsilon > 0 \\ \pi_1(X), & \text{if } \epsilon = 0. \end{cases}$$

Dendrograms over a given set can be bijectively associated with an ultrametric over the set. In this sense one can view the map $X \mapsto \theta_{\pi_1(X)}$ described in the theorem above as a map from Gromov-Hausdorff space into itself. We prove that this endomorphism is 2-Lipschitz.

Theorem 6.2 (d_{GH}-stability for $\theta_{\pi_1(\bullet)}$)**.** *If compact geodesic metric spaces X and Y are s.l.s.c., then*

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leqslant 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Of the three definitions of persistent fundamental groups given above, two of them can be generalized to the case of general homotopy groups thus giving rise to $\mathrm{P}\Pi_n^{\mathrm{K}}$ and $\mathrm{P}\Pi_n^{\mathrm{VR}}$ as persistent versions of π_n. As for the case of $n = 1$ described in Theorem 5.12, we still have that $\mathrm{P}\Pi_n^{\mathrm{K},\bullet} \cong \mathrm{P}\Pi_n^{\mathrm{VR},2\bullet}$, as well as the isomorphism of persistent homology groups $\mathrm{PH}_n^{\mathrm{K},\bullet} \cong \mathrm{PH}_n^{\mathrm{VR},2\bullet}$ (cf. Corollary 5.11). We furthermore have that under the interleaving distance (cf. Definition 3.8) the map from compact metric spaces into persistent groups $X \mapsto \mathrm{P}\Pi_n^{\mathrm{K}}(X)$ is 1-Lipschitz.

Theorem 6.1 (d_{I}-stability for $\mathrm{P}\Pi_n^{\mathrm{K}}(\bullet)$ and $\mathrm{P}\Pi_n^{\mathrm{VR}}(\bullet)$)**.** *Let (X, x_0) and (Y, y_0) be pointed compact metric spaces. Then, for each $n \in \mathbb{Z}_{\geqslant 1}$,*

$$d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{K}}(X, x_0), \mathrm{P}\Pi_n^{\mathrm{K}}(Y, y_0)\right) \leqslant d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

If X and Y are chain-connected, then

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_n^{\mathrm{K}}(X),\mathrm{P\Pi}_n^{\mathrm{K}}(Y)\right)\leqslant d_{\mathrm{GH}}(X,Y).$$

Via the isomorphism $\mathrm{P\Pi}_n^{\mathrm{K},\bullet}\cong\mathrm{P\Pi}_n^{\mathrm{VR},2\bullet}$, the above two inequalities also hold for $\mathrm{P\Pi}_n^{\mathrm{VR}}(\bullet)$, up to a factor $\frac{1}{2}$.

We relate persistent fundamental groups to the standard persistent homology groups via a suitable version of the Hurewicz Theorem.

Theorem 5.31 (Persistent Hurewicz Theorem)**.** *Let X be a chain-connected metric space. Then there is a natural transformation*

$$\mathrm{P\Pi}_1^{\mathrm{K}}(X)\overset{\rho}{\Rightarrow}\mathrm{PH}_1^{\mathrm{K}}(X),$$

where for each $\epsilon>0$, ρ_ϵ is surjective and $\ker(\rho_\epsilon)$ is the commutator group of $\mathrm{P\Pi}_1^{\mathrm{K},\epsilon}(X)$.

The persistent fundamental group of a metric space X often contains more information than the corresponding first Kuratowski (or Vietoris-Rips) persistent homology group. In Example 6.6 from §6.1, we compare the torus $\mathbb{S}^1\times\mathbb{S}^1$ with the wedge sum $\mathbb{S}^1\vee\mathbb{S}^1$ and obtain that $\mathrm{PH}_1^{\mathrm{K}}(\mathbb{S}^1\times\mathbb{S}^1)\cong\mathrm{PH}_1^{\mathrm{K}}(\mathbb{S}^1\vee\mathbb{S}^1)$, but

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_1^{\mathrm{K}}\left(\mathbb{S}^1\times\mathbb{S}^1\right),\mathrm{P\Pi}_1^{\mathrm{K}}\left(\mathbb{S}^1\vee\mathbb{S}^1\right)\right)=\frac{\pi}{6}=\sup_{n\geqslant 0}d_{\mathrm{I}}\left(\mathrm{PH}_n^{\mathrm{K}}\left(\mathbb{S}^1\times\mathbb{S}^1\right),\mathrm{PH}_n^{\mathrm{K}}\left(\mathbb{S}^1\vee\mathbb{S}^1\right)\right),$$

where the supremum on the right hand side is attained for $n=2$.

1.4. **Organization of the paper.** In §2 we provide a notation key to facilitate reading the paper.

In §3 we review elements from category theory, metric geometry, and topology which will be used throughout the paper. In particular §3.4 provides the necessary background about persistent homology, the interleaving distance, and related concepts.

In §4 we review the concept of discrete fundamental groups from the work of Plaut and Wilkins [PW13] and Barcelo et al. [BCW14], and prove the discretization theorem (cf. Theorem 4.19), which states that the limit of discrete fundamental groups is the classical fundamental group for tame enough spaces. Moreover, §4.4 constructs a stable (under the Gromov-Hausdorff distance of metric spaces) pseudo-metric $\mu_X^{(1)}$ on the set $\mathcal{L}(X,x_0)$ of all discrete chains in a metric space X. When X has a discrete and bounded set of critical values (as defined by [Wil11, Definition 2.3.4]), the metric $\mu_X^{(1)}$ arises from a generalized subdendrogram over $\mathcal{L}(X,x_0)$.

In §5.1 the definitions of the persistent homotopy groups $\mathrm{P\Pi}_1(\bullet)$, $\mathrm{P\Pi}_n^{\mathrm{VR}}(\bullet)$ and $\mathrm{P\Pi}_n^{\mathrm{K}}(\bullet)$ are given, together with their main properties and the proof of Theorem 5.12. To handle the high-dimensional homotopy groups of spheres that appear in the calculation of $\mathrm{P\Pi}_n^{\mathrm{K}}(\mathbb{S}^1)$, we tensor $\mathrm{P\Pi}_n^{\mathrm{K}}(\mathbb{S}^1)$ with rational numbers $\mathbb{Q}$ to generate computable rational persistent homotopy groups. In §5.2, we prove Theorem 5.15, and show that the ultrametric on $\pi_1(X)$ associated to the dendrogram $\theta_{\pi_1(X)}$ is stable when the underlying spaces are homotopy equivalent, cf. Theorem 5.19, together with an application on Riemannian manifolds. In §5.3 (resp. §5.4), we see that persistent fundamental groups under products (resp. wedge sums) of metric spaces are products (resp. coproducts) of persistent fundamental groups. In §5.5, a persistent version of Hurewicz theorem is proved, cf. Theorem 5.31.

In §6 we address the question of stability of persistent homotopy groups and prove Theorems 6.2 and 6.1. As an application of the stability results, we study three pairs of metric spaces: $\mathbb{S}^1 \times \mathbb{S}^1$ vs. $\mathbb{S}^1 \vee \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^m$ vs. $\mathbb{S}^1 \vee \mathbb{S}^m$ and $\mathbb{S}^1 \times \mathbb{S}^1$ vs. $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$, where $\mathbb{S}^m$ is the geodesic unit m-sphere. In the examples involving m-spheres for $m > 1$, $\mathrm{P}\Pi_m^{\mathrm{K}}$ (or $\mathrm{PH}_m^{\mathrm{K}}$) are restricted to some interval I_m. While $\mathrm{PH}_m^{\mathrm{K}}|_{I_m}$ fails to separate the two spaces in each case, $\mathrm{P}\Pi_m^{\mathrm{K}}|_{I_m}$ provides a positive lower bound for the Gromov-Hausdorff distance. In §6.2, an alternative proof, more constructive and independent of the notion of persistent K-fundamental groups, of the case $n = 1$ in Theorem 6.1 is given.

In §7, we apply stability theorems given in §6 to finite metrics spaces, and give examples of the generalized subdendrograms obtained in §4. We first prove that finite tree metric spaces have the trivial persistent fundamental group. In §7.1, we construct the generalized subdendrograms of cycle graphs C_m and star graphs S_n, where the graphs are endowed with the shortest-path distance. We compute $d_{\mathrm{GH}}(C_m, S_n)$ as a function of m, which grows at the same rate with $\frac{1}{4}m$. Meanwhile, the interleaving distance between $\mathrm{P}\Pi_1(C_m)$ and $\mathrm{P}\Pi_1(S_n)$ grows at rate of $\frac{1}{12}m$, which provides a fair approximation of $d_{\mathrm{GH}}(C_m, S_n)$ and is much easier to calculate. In §7.2, we study the finite metrics spaces $\frac{2\pi}{3}C_3$ and $\frac{\pi}{2}C_4$ as metric subspaces of $\mathbb{S}^1$, by computing several types of distances constructed in this paper. In this particular example, $\mu_\bullet^{(1)}$ provides a slightly better lower bound of $d_{\mathrm{GH}}\left(\frac{2\pi}{3}C_3, \frac{\pi}{2}C_4\right)$ than the d_{I} between persistent fundamental (or 1st-homology) groups.

Acknowledgements. We thank Prof. C. Plaut for bringing to our attention his results [Pla07] on connecting the discrete fundamental group construction of [Wil11] to the Vietoris-Rips complex (cf. Theorem 5.12). We also thank Prof J. Jardine for sharing his work on persistent fundamental groupoids and Gunnar Carlsson for his suggestion that we consider (persistent) rational homotopy groups.

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2. Notation

Symbol	**Meaning**
$\Bbbk$	A field.
$\mathcal{C}$	A category.
$\mathbf{Top}^*$	The category of pointed compactly generated weakly Hausdorff topological spaces.
$\mathcal{V}_\Bbbk$	Free-functor from the category of sets **Set** to the category of vector spaces **Vec**.
$\lim \mathbb{A}$	Limit of a family of a diagram $\mathbb{A}$, page 9.
$\mathcal{M}^{(\mathrm{pt})}$	The set of (pointed) compact metric spaces, page 10 (12).
d_{H}^X	Hausdorff distance between subsets of the metric space X, page 11.
d_{GH}	Gromov-Hausdorff distance between pseudo-metric spaces, page 11.
$d_{\mathrm{GH}}^{\mathrm{pt}}$	Pointed Gromov-Hausdorff distance between pseudo-metric spaces, page 12.
$\mathbf{P}\mathcal{C}^{(T,\leqslant)}$	The category of functors from $(T, \leqslant)$ to $\mathcal{C}$, for $T \subset \mathbb{R}$, page 13.
$\cong$	Isometry of metric spaces, or Homotopy equivalence of topological spaces, or Isomorphism in $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$ or **Grp**.

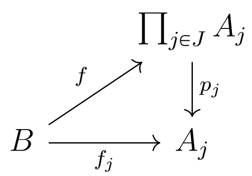
$\mathrm{Hom}(\mathbb{V},\mathbb{W})$	The collection of homorphisms from $\mathbb{V}$ to $\mathbb{W}$ in $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$, page 13.
$\mathrm{Hom}^{\delta}(\mathbb{V},\mathbb{W})$	The collection of homorphisms of degree δ from $\mathbb{V}$ to $\mathbb{W}$ in $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$, page 13.
d_{I}	Interleaving distance between functors, Definition 3.8.
d_{B}	bottleneck distance between multisets of points in $\bar{\mathbb{R}}^2$, page 14.
$\mathrm{VR}_{\bullet}(X)$	a filtration of X given by Vietoris–Rips complexes, page 15.
$\mathrm{PH}_k^{\mathrm{VR},\epsilon}(X)$	k-th (simplicial) homology group of $\mathrm{VR}_{\epsilon}(X)$ with coefficients in $\mathbb{Z}$, page 15.
$\mathrm{PH}_k^{\mathrm{VR}}(X)$	k-th persistent homology of X, page 15.
$\simeq$	Weak homotopy equivalence between topological spaces or $\mathbb{R}$-spaces, page 15.
d_{HI}	Homotopy interleaving distance between functors, Definition 3.11.
μ_{θ_A}	Ultrametric induced by the dendrogram θ_A on a set A, page 16.
$\mu^{\mathrm{s}}_{\theta_A}$	Pseudo-ultrametric induced by θ^{s}_A on a set A, page 17.
$\mathrm{P}\Pi_0(X)$	The persistent set induced by a metric space X on Page 17.
$[n]$	The set $\{0,\cdots,n-1\}$ for a non-negative set n, page 19.
(X,x_0)	A pointed metric space.
$\mathcal{L}(X,x_0)$	The set of discrete loops on (X,x_0), Proposition 4.9.
$\mathcal{L}^{\epsilon}(X,x_0)$	The set of ϵ-loops on (X,x_0), page 21.
$\alpha * \alpha'$	Concatenation of two discrete chains α and α', page 21.
$\pi_1^{\epsilon}(X,x_0)$	Discrete fundamental group at scale ϵ of (X,x_0), Definition 4.11.
$\pi_1^{\mathrm{E}}(K,v_0)$	Edge-path group of a simplicial complex K, page 22.
$\mathrm{Cr}(X)$	The set of critical values of a metric space X, page 25.
$\mathrm{P}\Pi_1(X,x_0)$	Persistent fundamental group of (X,x_0), Definition 5.2.
$\mathrm{P}\Pi_1^{\epsilon}(X,x_0)$	$\pi_1^{\epsilon}(X,x_0)$, page 22.
$A[I]$	Interval (generalized) persistence module, page 13.
X^{ϵ}	The ϵ-thickening of a compact metric space X, Definition 5.4.
$\mathrm{P}\Pi_n^{\mathrm{K}}(X,x_0)$	The n-th persistent K-homotopy group, Definition 5.5.
$\mathrm{PH}_n^{\mathrm{K}}(X)$	The n-th persistent K-homology group, Remark 5.6.
$\mathrm{P}\Pi_n^{\mathrm{VR}}(X,x_0)$	The n-th persistent VR-homotopy group, Definition 5.7.
$\mathrm{P}\Pi_1^{\mathrm{E}}(X,x_0)$	The persistent edge-path group of Vietoris-Rips complexes, page 28.

3. Background

3.1. **Category Theory.** Let $\Bbbk$ be a field. We denote the category of sets by **Set**, the category of vector spaces over $\Bbbk$ by **Vec**, and the category of groups by **Grp**. Also, let **Top** be the category of compactly generated weakly Hausdorff topological spaces [BL17], and let **Top*** be the category of pointed compactly generated weakly Hausdorff topological spaces.

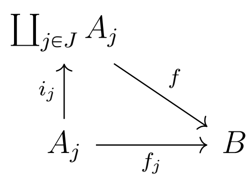
Let $\mathcal{V}_{\Bbbk}$ be the free-functor from **Set** to **Vec** defined by $\mathcal{V}_{\Bbbk} : A \mapsto \mathrm{Span}_{\Bbbk}(A)$, the linear space spanned by the set A over $\Bbbk$.

Now we introduce the definitions of product, coproduct and limit in a category $\mathcal{C}$. The *product* of a family of objects $(A_j)_{j\in J}$ indexed by a set J is an object $\prod_{j\in J} A_j$ of $\mathcal{C}$ together with morphisms $p_j : \prod_{j\in J} A_j \to A_j$ satisfying the following universal property: for any object B and J-indexed family of morphisms $f_j : B \to A_j$, there exists a unique morphism $f : B \to \prod_{j\in J} A_j$ such that the diagram



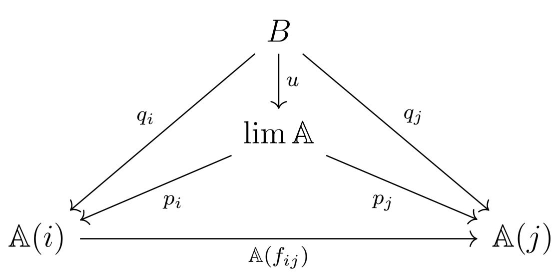
commutes for all $j \in J$.

The *coproduct* of a family of objects $(A_j)_{j\in J}$ indexed by a set J is an object $\coprod_{j\in J} A_j$ of $\mathcal{C}$ together with morphisms $i_j : A_j \to \coprod_{j\in J} A_j$ satisfying the following universal property: for any object B and J-indexed family of morphisms $f_j : A_j \to B$, there exists a unique morphism $f : \coprod_{j\in J} A_j \to B$ such that the diagram



commutes for all $j \in J$.

A *small category* is a category where the class of objects is a set. A functor from a small category J to a category $\mathcal{C}$ is called a J*-shaped diagram* in $\mathcal{C}$. Let J be a small category and let $\mathbb{A}$ be a J-shaped diagram. The *limit* of $\mathbb{A}$ is an object $\lim \mathbb{A}$ in $\mathcal{C}$ together with morphisms $p_j : \lim \mathbb{A} \to \mathbb{A}(j)$ for each $j \in J$ such that $p_j = \mathbb{A}(f_{ij}) \circ p_i$ for each map $f_{ij} : i \to j$ and the following universal property is satisfied: for any other $(B, (q_j)_{j\in J})$ such that $q_j = \mathbb{A}(f_{ij}) \circ q_i$ for each f_{ij}, there exists a unique morphism $u : B \to \lim \mathbb{A}$ such that the diagram



commutes for every map f_{ij}. For example, when $\mathcal{C} = \mathbf{Grp}$ the limit of a J-shaped diagram $\mathbb{A}$ is given by

$$\lim \mathbb{A} = \left\{ (a_j)_{j\in J} \in \prod_{j\in J} \mathbb{A}(j) : a_j = f_{ij}(a_i), \forall\, i \xrightarrow{f_{ij}} j \right\},$$

equipped with natural projections $p_j : \lim \mathbb{A} \to \mathbb{A}(j)$ by picking out the j-th component.

3.2. **Metric Spaces and General Topology.** Given a set X, an (extended) *pseudo-metric* d on X is a function $d : X \times X \to [0, +\infty]$ such that for any $x, y, z \in X$, the following holds:

- $d(x, x) = 0$;

- $d(x, y) = d(y, x)$;
- $d(x, z) \leqslant d(x, y) + d(y, z)$.

In this paper, the term pseudo-metric will be abused so that when the first condition is not satisfied, we still call it a pseudo-metric. A *metric* d on X is a pseudo-metric such that $d(x, y) = 0$ if and only if $x = y$. A *metric space* is a pair (X, d) where X is a set and d is a metric on X. A *(pseudo) ultrametric* d on X is a (pseudo) metric satisfying the strong version of the triangle inequality:

$$d(x, z) \leqslant \max\left\{d(x, y), d(y, z)\right\}, \forall x, y, z \in X.$$

A finite metric space (X, d_X) is called a *tree metric space*, if d_X satisfies the following condition (called the *four-point condition*):

$$d_X(x_1, x_3) + d_X(x_2, x_4) \leqslant \max\{d_X(x_1, x_2) + d_X(x_3, x_4), d_X(x_3, x_2) + d_X(x_1, x_4)\},$$

for any $x_1, x_2, x_3, x_4 \in X$. It is well-known that a metric space (X, d_X) is a tree metric space if and only if there is a tree T with non-negative edge lengths whose nodes contain X such that $d_X(x, y) = d_T(x, y)$, where $d_T(x, y)$ is the length of the shortest path between x and y.

For a finite pseudo-metric space, its metric can be represented by a symmetric square matrix containing the pairwise distance between elements, called the *distance matrix*. The metric space with exactly one point is called the *one-point metric space* and is denoted by $*$. Given two pseudo-metric spaces (X, d_X) and (Y, d_Y), a map $f : (X, d_X) \to (Y, d_Y)$ is called *distance-preserving* if $d_X(x, x') = d_Y(f(x), f(x'))$ for all $x, x' \in X$. A bijective distance-preserving map is called an *isometry*. Two pseudo-metric spaces are *isometric* and denoted by $X \cong Y$, if there exists an isometry between them.

Let (X, d_X) be a metric space. For $A \subset X$, we define the *diameter* of A to be

$$\mathrm{diam}(A) := \sup\{d_X(x, y) : x, y \in A\}.$$

If x and $\tilde{x}$ are two points in X such that $d_X(x, \tilde{x}) = \mathrm{diam}(X)$, then we say that $\tilde{x}$ is an *antipode* of x. Given $x \in X$, we define the *eccentricity* of x in X to be

$$\mathrm{ecc}(x) := \sup\{d_X(x, y) : y \in X\}.$$

The *radius* of X is defined to be

$$\mathrm{rad}(X) := \inf\{\mathrm{ecc}(x) : x \in X\}.$$

It is not hard to see that $\mathrm{diam}(X) = \sup\{\mathrm{ecc}(x) : x \in X\}$ and $\mathrm{rad}(X) \leqslant \mathrm{diam}(X)$. We call a point $x \in X$ a *center* of X, if $\mathrm{ecc}(x) = \mathrm{rad}(X)$. Centers of a metric space may not be unique. When X is compact, the existence of centers is guaranteed.

Let $\mathcal{M}$ be the collection of all compact metric spaces. For $(X, d_X) \in \mathcal{M}$, we denote by $B(x, r)$ the open ball of radius r centered at $x \in X$, and denote by A^r the *r-neighborhood* of a set A in X, i.e.,

$$A^r := \bigcup_{x \in A} B(x, r).$$

Let us recall the following definitions from Vigolo [Vig18]. As a topological space with the metric topology, X is *uniformly locally path connected* (u.l.p.c.) if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ any two points in $B(x, \delta)$ are connected by a path γ with image completely contained in $B(x, \epsilon)$. The space X is *uniformly semi-locally simply connected* (u.s.l.s.c.) if there exists $\epsilon > 0$ small enough so that for every $x \in X$, a loop with image contained in $B(x, \epsilon)$ is null-homotopic in X (but is not necessarily null-homotopic in $B(x, \epsilon)$).

We now review some properties of homology groups and homotopy groups from [Hat01]. Given two connected topological spaces X and Y, let $X \vee Y$ denote the wedge sum (cf. [Hat01, Page 10]) and let $X \times Y$ denote the product space. For each $m \geqslant 1$, denote by $\mathbb{S}^m$ the m-dimensional sphere $\mathbb{S}^m$.

Proposition 3.1. *Let $n \geqslant 1$. Then, the following is true when homology groups are computed with coefficients $\mathbb{Z}$ or $\mathbb{R}$.*

(1) $\mathrm{H}_n(X \vee Y) \cong \mathrm{H}_n(X) \oplus \mathrm{H}_n(Y)$*;*

(2) if X and Y are CW complexes, then there are natural short exact sequences

$$0 \to \bigoplus_{i+j=n} (\mathrm{H}_i(X) \otimes \mathrm{H}_j(Y)) \to \mathrm{H}_n(X \times Y) \to \bigoplus_{i+j=n} \mathrm{Tor}(\mathrm{H}_i(X), \mathrm{H}_{j-1}(Y)) \to 0$$

and these sequences split.

(3) $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$*;*

(4) $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$*, as a consequence of the van Kampen's theorem. For $n \geqslant 2$,*

(a) $\pi_n(\mathbb{S}^1 \vee \mathbb{S}^n) \cong \mathbb{Z}[t, t^{-1}]$ *the Laurent polynomials in t and t^{-1}.*

(b) Let $\mathbb{S}^n_\alpha := \mathbb{S}^n$ for each $\alpha \in A$ where A is a given set. Then $\pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$*.*

3.3. **Gromov-Hausdorff Distance.** We recall the definition of the Gromov-Hausdorff distance [BBB+01]. Let Z_1 and Z_2 be subspaces of a metric space (X, d). The *Hausdorff distance* between Z_1 and Z_2 is defined to be

$$d_{\mathrm{H}}^X(Z_1, Z_2) := \inf\left\{r > 0 : Z_1 \subset Z_2^r \text{ and } Z_2 \subset Z_1^r\right\}.$$

For metric spaces (X, d_X) and (Y, d_Y), the *Gromov-Hausdorff distance* between them is the infimum of $r > 0$ for which there exist a metric spaces Z and two distance preserving maps $\psi_X : X \to Z$ and $\psi_Y : Y \to Z$ such that $d_{\mathrm{H}}^Z(\psi_X(X), \psi_Y(Y)) < r$, i.e.,

$$d_{\mathrm{GH}}(X, Y) := \inf_{Z, \psi_X, \psi_Y} d_H^Z(\psi_X(X), \psi_Y(Y)).$$

Given two metric spaces (X, d_X) and (Y, d_Y), the *distortion* of a map $\varphi : X \to Y$ is given by

$$\mathrm{dis}(\varphi) := \sup_{x, x' \in X} |d_X(x, x') - d_Y(\varphi(x), \varphi(x'))|.$$

For maps $\varphi : X \to Y$ and $\psi : Y \to X$, their *co-distortion* is defined to be

$$\mathrm{codis}(\varphi, \psi) := \sup_{x \in X, y \in Y} |d_X(x, \psi(y)) - d_Y(\varphi(x), y)|.$$

Theorem 3.2 (Theorem 2.1, [KO97])**.** *For two bounded metric spaces (X, d_X) and (Y, d_Y),*

$$d_{\mathrm{GH}}(X, Y) = \inf_{\substack{\varphi : X \to Y \\ \psi : Y \to X}} \tfrac{1}{2} \max\{\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi, \psi)\}.$$

A *tripod* between two sets X and Y is a pair of surjections from another set Z to X and Y respectively. We will express this by the diagram

$$R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y.$$

For $x \in X$ and $y \in Y$, by $(x, y) \in R$ we mean there exists $z \in Z$ such that $\phi_X(z) = x$ and $\phi_Y(z) = y$. When (X, d_X) and (Y, d_Y) are pseudo-metric spaces, the *distortion* of a tripod

R between X and Y is defined to be:

$$\mathrm{dis}(R) := \sup_{z,z' \in Z} \left| d_X(\phi_X(z), \phi_X(z')) - d_Y(\phi_Y(z), \phi_Y(z')) \right|.$$

Let $\mathfrak{R}(X, Y)$ denote the collection of all tripods between X and Y. Clearly, a tripod R satisfies the following property: for any $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in R$ and for any $y \in Y$ there exists at least one $x \in X$ such that $(x, y) \in R$.

Theorem 3.3 (Theorem 7.3.25, [BBB+01])**.** *For any two bounded metric spaces X and Y,*

$$d_{\mathrm{GH}}(X, Y) = \tfrac{1}{2} \inf_{R \in \mathfrak{R}(X,Y)} \mathrm{dis}(R).$$

Remark 3.4. In this theorem, the formula on the right-hand side applies to pseudo-metric spaces (see [CM17, CM18b]) as a generalization of the Gromov-Hausdorff distance. For this reason, we still use the symbol d_{GH} to denote this generalized distance.

Proposition 3.5 (p. 255, [BBB+01])**.** *For bounded metric spaces (X, d_X) and (Y, d_Y),*

$$\tfrac{1}{2} |\,\mathrm{diam}(X) - \mathrm{diam}(Y)| \leqslant d_{\mathrm{GH}}(X, Y) \leqslant \tfrac{1}{2} \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}.$$

In particular, if Y is the one-point metric space $*$, then

$$d_{\mathrm{GH}}(X, *) = \tfrac{1}{2}\, \mathrm{diam}(X).$$

The lower bound in Proposition 3.5 can be improved in the following case:

Proposition 3.6. *Suppose $(X, d_X) \in \mathcal{M}$ is such that every $x \in X$ has an antipode. Let $(Y, d_Y) \in \mathcal{M}$ be such that $\mathrm{rad}(Y) \leqslant \mathrm{diam}(X)$. Then*

$$\tfrac{1}{2} \left(\mathrm{diam}(X) - \mathrm{rad}(Y)\right) \leqslant d_{\mathrm{GH}}(X, Y). \tag{1}$$

Proof. Let y_0 be a center of Y. Suppose R is an arbitrary tripod between X and Y. Clearly, there exists some $x \in X$ such that $(x, y_0) \in R$. Let $\tilde{x}$ be an antipode of x, i.e., $d_X(x, \tilde{x}) = \mathrm{diam}(X)$. There exists some $y \in Y$ such that $(\tilde{x}, y) \in R$. Therefore,

$$\mathrm{diam}(X) - \mathrm{rad}(Y) \leqslant d_X(x, \tilde{x}) - d_Y(y_0, y) \leqslant \mathrm{dis}(R).$$

Since R is arbitrary, we can conclude that Equation (1) is true. ∎

A *pointed metric space* (X, x_0, d_X) is a metric space (X, d_X) together with a distinguished basepoint $x_0 \in X$. For the sake of simplicity, we will omit the distance function d_X and denote a pointed metric space by (X, x_0). Let $\mathcal{M}^{\mathrm{pt}} = \{(X, x_0) | x_0 \in X, (X, d_X) \in \mathcal{M}\}$ be the collection of all pointed compact metric spaces. Given basepoints $x_0 \in X$ and $y_0 \in Y$, a *pointed tripod* is a tripod $R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ such that $\phi_X^{-1}(x_0) \cap \phi_Y^{-1}(y_0)$ is non-empty. Let $\mathfrak{R}^{\mathrm{pt}}((X, x_0), (Y, y_0))$ denote the collection of pointed tripods between (X, x_0) and (Y, y_0). The *pointed Gromov-Hausdorff distance* between X and Y is defined to be

$$d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)) := \tfrac{1}{2} \inf_{R \in \mathfrak{R}^{\mathrm{pt}}((X,x_0),(Y,y_0))} \mathrm{dis}(R).$$

It is clear that

$$d_{\mathrm{GH}}(X, Y) = \inf_{x_0 \in X,\, y_0 \in Y} d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

3.4. **Persistence Theory.** We recall the definitions of persistence modules and the interleaving distance, as well as the stability theorem for the interleaving distance and the bottleneck distance, from [BL17, Oud15].

Let $T = \mathbb{R}_+, \mathbb{R}_{\geqslant 0}$ or $\mathbb{R}$. Note that in any of the three cases there is a canonical poset structure on T, denoted by $(T, \leqslant)$. Consequently, $(T, \leqslant)$ can be viewed as a category whose objects are real numbers and the set of morphisms from an object a to an object b consists of a single morphism if $a \leqslant b$ and is otherwise empty. Let $\mathcal{C}$ be any arbitrary category. The collections of functors

$$\mathbb{V} : (T, \leqslant) \to \mathcal{C},$$

and natural transformations between functors forms a category denoted by $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$ (see [BS14, §2]). Recall that given two objects $\mathbb{V}$ and $\mathbb{W}$ of $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$, a natural transformation from $\mathbb{V}$ to $\mathbb{W}$

$$f : \mathbb{V} \Rightarrow \mathbb{W},$$

also called a *homomorphism* from $\mathbb{V}$ to $\mathbb{W}$, is a family of morphisms in $\mathcal{C}$: $\{f_t : V_t \to W_t\}_{t\in T}$ such that the diagram

$$\begin{array}{ccc} V_t & \xrightarrow{v_{t,t'}} & V_{t'} \\ {\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle f_{t'}} \\ W_t & \xrightarrow[w_{t,t'}]{} & W_{t'}. \end{array}$$

commutes for $t \leqslant t'$. If f_t is an isomorphism in $\mathcal{C}$ for each $t \in T$, then the homomorphism f is called a *(natural) isomorphism* between $\mathbb{V}$ and $\mathbb{W}$, in which case we write $\mathbb{V} \cong \mathbb{W}$. The collection of homomorphisms from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$.

For an object $A \in \mathcal{C}$ and an interval $I \subset T$, the *interval (generalized) persistence module* $A[I]$ is defined as follows: $A[I] = \mathbb{0}$ if I is empty and otherwise,

$$(A[I])\,(t) = \begin{cases} A, & \text{if } t \in I, \\ 0, & \text{otherwise,} \end{cases}$$

where $(A[I])(s \leqslant t) = \mathrm{Id}_A$ when s and t are both in I, and $(A[I])(s \leqslant t) = 0$ otherwise.

Let $\delta \geqslant 0$. We define $S_\delta : (T, \leqslant) \to (T, \leqslant)$ to be the functor given by $S_\delta(t) = t + \delta$ and define $\eta_\delta : \mathrm{Id}_{(T,\leqslant)} \Rightarrow S_\delta$ to be the natural transformation given by $\eta_\delta(t) : t \leqslant t + \delta$. Note that $S_\delta \circ S_{\delta'} = S_{\delta+\delta'}$ and $\eta_\delta \circ \eta_{\delta'} = \eta_{\delta+\delta'}$. A *homomorphism of degree* δ from $\mathbb{V}$ to $\mathbb{W}$ is a natural transformation

$$f : \mathbb{V} \Rightarrow \mathbb{W} \circ S_\delta.$$

The collection of homorphisms of degree δ from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\mathrm{Hom}^\delta(\mathbb{V}, \mathbb{W})$.

In this paper, we will consider several choices of $\mathcal{C}$:

- $\mathcal{C} = \mathbf{Set}$ yields $\mathbf{PSet}^{(T,\leqslant)}$, whose objects are called *persistent sets*;
- $\mathcal{C} = \mathbf{Vec}$ yields $\mathbf{PVec}^{(T,\leqslant)}$, whose objects are called *persistence modules*;
- $\mathcal{C} = \mathbf{Grp}$ yields $\mathbf{PGrp}^{(T,\leqslant)}$, whose objects are called *persistent groups*;
- $\mathcal{C} = \mathbf{Top}$ yields $\mathbf{PTop}^{(T,\leqslant)}$, whose objects are called *T-spaces.*

Definition 3.7 (Definition 3.1, [BS14])**.** An δ-interleaving of $\mathbb{V}$ and $\mathbb{W}$ consists of natural transformations $f : \mathbb{V} \Rightarrow \mathbb{W} \circ S_\delta$ and $g : \mathbb{W} \Rightarrow \mathbb{V} \circ S_\delta$ (i.e., $f \in \mathrm{Hom}^\delta(\mathbb{V}, \mathbb{W})$ and $g \in \mathrm{Hom}^\delta(\mathbb{W}, \mathbb{V})$) such that the following diagrams commute for all $t \in T$:

$$(gS_\delta)f = \mathbb{V}\eta_{2\delta} \text{ and } (fS_\delta)g = \mathbb{W}\eta_{2\delta}.$$

If there exists a δ-interleaving $(\mathbb{V}, \mathbb{W}, f, g)$, then we say that $\mathbb{V}$ and $\mathbb{W}$ are δ-*interleaved.*

In other words, a δ-interleaving $(\mathbb{V}, \mathbb{W}, f, g)$ consists of families of morphisms $\{f_t : V_t \to W_{t+\delta}\}_{t\in T}$ and $\{g_t : W_t \to V_{t+\delta}\}_{t\in T}$ such that the following diagrams commute for all $t \leqslant t'$:

$$\begin{tikzcd} V_t \arrow[r, "v_{t,t'}"] \arrow[dr, "f_t"'] & V_{t'} \arrow[dr, "f_{t'}"] & \\ & W_{t+\delta} \arrow[r, "w_{t+\delta,t'+\delta}"'] & W_{t'+\delta} \end{tikzcd} \qquad \begin{tikzcd} & V_{t+\delta} \arrow[r, "v_{t+\delta,t'+\delta}"] & V_{t'+\delta} \\ W_t \arrow[ur, "g_t"] \arrow[r, "w_{t,t'}"'] & W_{t'} \arrow[ur, "g_{t'}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} V_t \arrow[rr, "v_{t,t+2\delta}"] \arrow[dr, "f_t"'] & & V_{t+2\delta} \\ & W_{t+\delta} \arrow[ur, "g_{t+\delta}"'] & \end{tikzcd} \qquad \begin{tikzcd} & V_{t+\delta} \arrow[dr, "f_{t+\delta}"] & \\ W_t \arrow[ur, "g_t"] \arrow[rr, "w_{t,t+2\delta}"'] & & W_{t+2\delta}. \end{tikzcd}$$

Definition 3.8. Let $\mathbb{V}$ and $\mathbb{W}$ be objects of $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$. The *interleaving distance* between $\mathbb{V}$ and $\mathbb{W}$ is

$$d_{\mathrm{I}}(\mathbb{V}, \mathbb{W}) := \inf\{\delta \geqslant 0 : \mathbb{V} \text{ and } \mathbb{W} \text{ are } \delta\text{-interleaved}\}.$$

Here we follow the convention that $\inf \varnothing = +\infty$. A quick fact is that d_{I} descends to a distance on isomorphism classes of objects in $\mathbf{P}\mathcal{C}^{(T,\leqslant)}$.

Persistence Modules and Persistence Diagrams. Let $\mathcal{C} = \mathbf{Vec}$ and let $T = \mathbb{R}_+$, unless otherwise specified. Recall from [Oud15] that a persistence module $\mathbb{V}$ is q-*tame* if $\mathrm{rank}(v_{t,t'} : V_t \to V_{t'}) < \infty$ whenever $t < t'$. If a persistence module $\mathbb{V}$ can be decomposed as a direct sum of interval modules (e.g. $\mathbb{V}$ is q-tame), say $\mathbb{V} \cong \bigoplus_{l\in L} \Bbbk(p_l^*, q_l^*)$ where $*$ indicates whether the interval is half-open or not (see [CdSGO16]), then its *(undecorated) persistence diagram* is the multiset

$$\mathrm{dgm}(\mathbb{V}) := \{(p_l, q_l) : l \in L\} - \Delta,$$

where $\Delta := \{(r, r) : r \in \mathbb{R}\}$ is the diagonal in the real plane.

The *bottleneck distance* between persistence diagrams, and more generally between multisets A and B of points in $\bar{\mathbb{R}}^2$, where $\bar{\mathbb{R}}$ are the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$, is defined as follows:

$$d_{\mathrm{B}}(A, B) := \inf \left\{ \sup_{a\in A} \|a - \phi(a)\|_\infty : \phi : A \cup \Delta^\infty \to B \cup \Delta^\infty \text{ a bijection } \right\}.$$

Here $\|(p, q) - (p', q')\|_\infty := \max\{|p - p'|, |q - q'|\}$ for each $p, q, p', q' \in \mathbb{R}$ and Δ^∞ is the multiset consisting of each point on the diagonal $\{(r, r) : r \in \bar{\mathbb{R}}\}$ in the extended real plane, taken with infinite multiplicity (see [CM18a]).

Theorem 3.9 (Theorem 5.14, [CdSGO16])**.** *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be* q*-tame persistence modules. Then,*

$$d_{\mathrm{I}}(\mathbb{V}, \mathbb{W}) = d_{\mathrm{B}}(\mathrm{dgm}(\mathbb{V}), \mathrm{dgm}(\mathbb{W})),$$

where $d_{\mathrm{I}}(\mathbb{V}, \mathbb{W})$ *is defined in Definition 3.8 with* $\mathcal{C} = \mathbf{Vec}$*.*

Vietoris-Rips Complexes. By first constructing a simplicial filtration out of a metric space and then applying the simplicial homology functor, one obtains a persistence module which encodes computable invariants of the original space. In this paper, we focus on the simplicial filtration given by Vietoris-Rips complexes, and recall the definitions from [AA17]. Suppose that (X, d_X) is a compact metric space and ϵ is a non-negative real number. The *Vietoris–Rips complex* $\mathrm{VR}_\epsilon(X)$ is the simplicial complex with vertex set X, where

$$\text{a finite subset } \sigma \subset X \text{ is a face of } \mathrm{VR}_\epsilon(X) \text{ iff } \mathrm{diam}(\sigma) \leqslant \epsilon.$$

The collection $\{\mathrm{VR}_\epsilon(X)\}_{\epsilon \geqslant 0}$ together with the natural simplicial inclusions forms a filtration of X, denoted by $\mathrm{VR}_\bullet(X)$.

Let G be an Abelian group. Given $k \in \mathbb{Z}_{\geqslant 0}$ and $\epsilon \geqslant 0$, let $C_k(\mathrm{VR}_\epsilon(X); G)$ be the Abelian group consisting of chains of the form $\sum_i n_i \sigma_i$ where coefficients n_i are taken from G and each σ_i is a k-simplex in $\mathrm{VR}_\epsilon(X)$. Let $\partial_k : C_k(\mathrm{VR}_\epsilon(X); G) \to C_{k-1}(\mathrm{VR}_\epsilon(X); G)$ be the k-th boundary operator. We denote by

$$\mathrm{PH}_k^{\mathrm{VR},\epsilon}(X; G) := \mathrm{H}_k(\mathrm{VR}_\epsilon(X); G)$$

the k-th (simplicial) homology group of $\mathrm{VR}_\epsilon(X)$ with coefficients in G. Then $\{\mathrm{PH}_k^{\mathrm{VR},\epsilon}(X; G)\}_{\epsilon \geqslant 0}$ together with the maps induced by natural inclusions forms an object in $\mathbf{PGrp}^{(\mathbb{R}_{\geqslant 0}, \leqslant)}$, denoted by $\mathrm{PH}_k^{\mathrm{VR}}(X; G)$. The collection of these modules for k ranging over all dimensions is called the *persistent homology of* X and denoted by $\mathrm{PH}_1^{\mathrm{VR}}(X; G)$. We will omit the coefficient group when $G = \mathbb{Z}$. When $G = \Bbbk$ is a field, $\mathrm{PH}_k^{\mathrm{VR}}(X; \Bbbk)$ is a persistent module and it is shown in [CdSO14] that if X is compact, then $\mathrm{PH}_k^{\mathrm{VR}}(X; \Bbbk)$ is q-tame for all $k \in \mathbb{Z}_{\geqslant 0}$. The persistence diagram corresponding to $\mathrm{PH}_k^{\mathrm{VR}}(X; \Bbbk)$ is denoted by $\mathrm{dgm}_k(X)$ for each $k \in \mathbb{Z}_{\geqslant 0}$.

Theorem 3.10 (Stability Theorem for d_B, [CCSG+09, CdSO14])**.** *Let* (X, d_X) *and* (Y, d_Y) *be from* $\mathcal{M}$*. Then, for any* $k \in \mathbb{Z}_{\geqslant 0}$,

$$d_\mathrm{I}(\mathrm{PH}_k^{\mathrm{VR}}(X; \Bbbk), \mathrm{PH}_k^{\mathrm{VR}}(Y; \Bbbk)) = d_\mathrm{B}(\mathrm{dgm}_k(X), \mathrm{dgm}_k(Y)) \leqslant 2 \cdot d_\mathrm{GH}(X, Y).$$

$\mathbb{R}$-spaces and Homotopy Interleavings. Recall that **Top** is the category of compactly generated weakly Hausdorff topological spaces, and an object in $\mathbf{PTop}^{(\mathbb{R}, \leqslant)}$ is also called an $\mathbb{R}$-*space* (see [BL17]). Let $\mathbb{X}$ be an $\mathbb{R}$-space. If for every $t \leqslant s \in \mathbb{R}$, the map $X_t \to X_s$ is an inclusion, then we call $\mathbb{X}$ a *filtration*. Blumberg and Lesnick showed that the interleaving distance between $\mathbb{R}$-spaces is not homotopy invariant (see [BL17, Remark 3.3]), and then defined homotopy interleavings between $\mathbb{R}$-spaces as certain homotopy-invariant analogues of interleavings.

Given two $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$, a natural transformation $f : \mathbb{X} \Rightarrow \mathbb{Y}$ is an *(objectwise) weak equivalence* if for each $t \in \mathbb{R}$, $f_t : X_t \to Y_t$ is a weak homotopy equivalence, i.e., it induces isomorphisms on all homotopy groups. The $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ are *weakly equivalent*, denoted by $\mathbb{X} \simeq \mathbb{Y}$, if there exists an $\mathbb{R}$-space $\mathbb{W}$ and natural transformations $f : \mathbb{W} \Rightarrow \mathbb{X}$ and $g : \mathbb{W} \Rightarrow \mathbb{Y}$ that are (objectwise) weak equivalences:

$$\mathbb{X} \overset{f}{\Leftarrow} \mathbb{W} \overset{g}{\Rightarrow} \mathbb{Y}.$$

The relation $\mathbb{X} \simeq \mathbb{Y}$ is an equivalence relation (see [BL17] for details). Given $T \subset \mathbb{R}$, we can consider the restriction of any given $\mathbb{R}$-space to T and study T-spaces in a similar way. For $\delta \geqslant 0$, two $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ are *δ-homotopy-interleaved* if there exist $\mathbb{R}$-spaces $\mathbb{X}' \simeq \mathbb{X}$ and $\mathbb{Y}' \simeq \mathbb{Y}$ such that $\mathbb{X}'$ and $\mathbb{Y}'$ are δ-interleaved, as in Definition 3.7 with $\mathcal{C} = \mathbf{Top}$.

Definition 3.11 (Homotopy interleaving distance)**.** The *homotopy interleaving distance* between two $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ is given by

$$d_\mathrm{HI}(\mathbb{X}, \mathbb{Y}) := \inf \{\delta \geqslant 0 : \mathbb{X}, \mathbb{Y} \text{ are } \delta\text{-homotopy-interleaved}\}.$$

Theorem 3.12 (Theorem 1.6, [BL17]). *Given two compact metric spaces X and Y, we have*

$$d_{\mathrm{B}}(\mathrm{dgm}_k(X), \mathrm{dgm}_k(Y)) \leqslant d_{\mathrm{HI}}(|\mathrm{VR}_\bullet(X)|, |\mathrm{VR}_\bullet(Y)|) \leqslant 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Here $|\mathrm{VR}_\bullet(X)|$ denotes the $\mathbb{R}_{\geqslant 0}$-space given by the geometric realizations of Vietoris-Rips complexes of X.

3.5. **Dendrograms and Generalized Subdendrograms.** For this section, we refer to [CM10]. Let A be a set, and let $\mathrm{Part}(A)$ be the set of partitions of A.

Definition 3.13. A *dendrogram* over A is a pair (A, θ_A), where $\theta_A : \mathbb{R}_{\geqslant 0} \to \mathrm{Part}(A)$ satisfies:

1. If $t \leqslant s$, then $\theta_A(t)$ refines $\theta_A(s)$.
2. For all r there exists $\delta > 0$ s.t. $\theta_A(r) = \theta_A(t)$ for all $t \in [r, r + \delta]$.
3. There exists t_0 such that $\theta_A(t)$ is the single block partition for all $t \geqslant t_0$.
4. $\theta_A(0)$ is the partition into singletons.

The following function on $A \times A$ defines an ultrametric on A (see [CM10]): for any $x, x' \in A$,

$$\mu_{\theta_A}(x, x') := \min\{\delta : x \text{ and } x' \text{ belong to the same block of } \theta_A(\delta)\}.$$

Let (A, θ_A) and (B, θ_B) be two dendrograms. Given $\delta \geqslant 0$, we say that the set maps $\phi : A \to B$ and $\psi : B \to A$ provide a *δ-interleaving* between θ_A and θ_B iff for all $x \in A, y \in B$ and $t \geqslant 0$,

$$\phi(\langle x \rangle_t^A) \subset \langle \phi(x) \rangle_{t+\delta}^B \text{ and } \psi(\langle y \rangle_t^B) \subset \langle \psi(y) \rangle_{t+\delta}^A,$$

where $\langle x \rangle_t^A$ represents the subset of A containing x in the partition $\theta_A(t)$. If such a δ-interleaving exists, we say that θ_A and θ_B are δ-interleaved. The *interleaving distance* between dendrograms θ_A and θ_B is defined to be

$$d_{\mathrm{I}}(\theta_A, \theta_B) := \inf\{\delta > 0 : A \text{ and } B \text{ are } \delta\text{-interleaved}\}.$$

Remark 3.14. Notice that each dendrogram θ_A can be viewed as an element of **PSet**, following from the definition of θ_A and the fact that $\mathrm{Part}(A)$ is a subcategory of **Set**. Using Definition 3.8, we can construct the interleaving distance, denoted by $d_{\mathrm{I}}^{\mathbf{Set}}$, between dendrograms (A, θ_A) and (B, θ_B). It turns out that $d_{\mathrm{I}}(\theta_A, \theta_B) = d_{\mathrm{I}}^{\mathbf{Set}}(\theta_A, \theta_B)$. Indeed, each δ-interleaving $(\phi : A \to B, \psi : B \to A)$ clearly induces a **PSet**-δ-interleaving. Conversely, given a **PSet**-δ-interleaving $(\Phi : \mathrm{Part}(A) \to \mathrm{Part}(B), \Psi : \mathrm{Part}(B) \to (A))$, we can define map $\phi : A \to B$ with $a \mapsto b \in \Phi(0)(\{a\})$, and similarly define $\psi : B \to A$. Although ϕ and ψ are not uniquely defined, it is not hard to see that the resulting (ϕ, ψ) is always a δ-interleaving regardless of the choices.

The functor $\mathcal{V}_{\Bbbk} \circ \theta_A : (\mathbb{R}_+, \leqslant) \to \mathbf{Vec}$, with $\mathcal{V}_{\Bbbk} \circ \theta_A(t) = \mathcal{V}_{\Bbbk}(\theta_A(t))$ and $\mathcal{V}_{\Bbbk} \circ \theta_A(t \leqslant s) = \mathcal{V}_{\Bbbk}(\theta_A(t)) \to \mathcal{V}_{\Bbbk}(\theta_A(s))$ induced by refinement of partitions, is then a persistence module.

Proposition 3.15 (§3.5, [CM10]). *Let (A, θ_A) and (B, θ_B) be two dendrograms. Then,*

$$\tfrac{1}{2} \cdot d_{\mathrm{I}}(\mathcal{V}_{\Bbbk} \circ \theta_A, \mathcal{V}_{\Bbbk} \circ \theta_B) \leqslant d_{\mathrm{GH}}((A, \mu_{\theta_A}), (B, \mu_{\theta_B})) \leqslant d_{\mathrm{I}}(\theta_A, \theta_B).$$

Proof. The first inequality follows from an argument in [CM10]. Assume $d_{\mathrm{GH}}((X, \mu_{\theta_A}), (Y, \mu_{\theta_B})) \leqslant \delta/2$ for some $\delta \geqslant 0$. Then there are maps $\phi : A \to B$ and $\psi : B \to A$ such that

- if x and x' are in the same block of $\theta_A(t)$, then $\phi(x)$ and $\phi(x')$ belong to the same block of $\theta_B(t + \delta)$;
- if y and y' are in the same block of $\theta_B(t)$, then $\psi(y)$ and $\psi(y')$ belong to the same block of $\theta_A(t + \delta)$.

In other words, ϕ and ψ induce homomorphisms of degree δ on persistence modules $\mathcal{V}_{\Bbbk} \circ \theta_A$ and $\mathcal{V}_{\Bbbk} \circ \theta_B$, denoted by ϕ_δ and ψ_δ, respectively. Given $t > 0$, if x and x' fall into the same block of $\theta_A(t)$, then $\psi \circ \phi(x), \psi \circ \phi(x')$ belong to the same block of $\theta_A(t + 2\delta)$. Thus, $\psi_\delta \circ \phi_\delta = 1^{2\delta}_{\mathcal{V}_{\Bbbk} \circ \theta_A}$. Similarly, we have $\phi_\delta \circ \psi_\delta = 1^{2\delta}_{\mathcal{V}_{\Bbbk} \circ \theta_B}$. Therefore, $\mathcal{V}_{\Bbbk} \circ \theta_A$ and $\mathcal{V}_{\Bbbk} \circ \theta_B$ are δ-interleaved. Letting $\delta \searrow 2d_{\mathrm{GH}}((A, \mu_{\theta_A}), (B, \mu_{\theta_B}))$, the first inequality follows.

Suppose (ϕ, ψ) is a δ-interleaving between θ_A and θ_B. Clearly,

$$R : A \xleftarrow{(\mathrm{Id}_A, \phi)} A \sqcup B \xrightarrow{(\psi, \mathrm{Id}_B)} B$$

forms a tripod between A and B, with distortion no larger than 2δ. Therefore, the second inequality is true. □

Given two sets A and B, let $P_A := \{A_1, \cdots, A_k\}$ and $P_B := \{B_1, \cdots, B_l\}$ be partitions of A and B, respectively. We define the product of P_A and P_B to be

$$P_A \times P_B := \{(A_i, B_j) : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l\},$$

which is clearly a partition of $A \times B$. In fact, the map $(P_A, P_B) \mapsto P_A \times P_B$ gives an embedding $\mathrm{Part}(A) \times \mathrm{Part}(B) \hookrightarrow \mathrm{Part}(A \times B)$. For two dendrograms (A, θ_A) and (B, θ_B), we define their product to be $(A \times B, \theta_A \times \theta_B)$, where

$$\theta_A \times \theta_B : \mathbb{R}_{\geqslant 0} \to \mathrm{Part}(A \times B) \text{ with } t \mapsto \theta_A(t) \times \theta_B(t).$$

It follows directly from Definition 3.13 that $(A \times B, \theta_A \times \theta_B)$ forms a dendrogram.

A *subpartition* of a set A is a partition of one of its subsets $A' \subset A$. Let $\mathrm{SubPart}(A)$ be the set of subpartitions of A.

Definition 3.16. A *generalized subdendrogram* of A is a pair $(A, \theta^{\mathrm{s}}_A)$, where $\theta^{\mathrm{s}}_A : \mathbb{R}_+ \to \mathrm{SubPart}(A)$ satisfies:

(1') If $t \leqslant s$, then $\theta^{\mathrm{s}}_A(t)$ and $\theta^{\mathrm{s}}_A(s)$ are partitions of A_t and A_s, respectively, where $A_t \subset A_s \subset A$ and $\theta^{\mathrm{s}}_A(t)$ refines $\theta^{\mathrm{s}}_A(s)|_{A_t} := \{B \cap A_t : B \in \theta^{\mathrm{s}}_A(s)\}$.
(2) For all r there exists $\delta > 0$ s.t. $\theta^{\mathrm{s}}_A(r) = \theta^{\mathrm{s}}_A(t)$ for all $t \in [r, r + \delta]$.
(3) There exists t_0 such that $\theta^{\mathrm{s}}_A(t)$ is the single block partition for all $t \geqslant t_0$.

Similarly, we consider the following function on $A \times A$: for any x and x' in A,

$$\mu^{\mathrm{s}}_{\theta_A}(x, x') := \min\{\delta \geqslant 0 : x, x' \text{ belong to the same block of } \theta_A(\delta)\},$$

which turns out to be a pseudo-ultrametric.

Proposition 3.17. *Let $(A, \theta^{\mathrm{s}}_A)$ and $(B, \theta^{\mathrm{s}}_B)$ be two generalized subdendrograms. Then*

$$\tfrac{1}{2} \cdot d_{\mathrm{I}}(\mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_A, \mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_B) \leqslant d_{\mathrm{GH}}((A, \mu^{\mathrm{s}}_{\theta_A}), (B, \mu^{\mathrm{s}}_{\theta_B})).$$

Proof. This follows from a similar proof with Proposition 3.15. □

4. Discrete Fundamental Groups

4.1. **Persistent Sets.** Given a compact metric space (X, d_X) and $\epsilon \geqslant 0$, let $\pi^\epsilon_0(X) = X/\sim^\epsilon_0$, where $x \sim^\epsilon_0 x'$ if there exists $\{x_0, \cdots, x_n\} \subset X$ such that $x_0 = x, x_n = x'$ and $d_X(x_i, x_{i+1}) \leqslant \epsilon$. For simplicity, we will write the sequence $\{x_0, \cdots, x_n\}$ as $x_0 \cdots x_n$. For $\epsilon' \geqslant \epsilon \geqslant 0$, there is a natural map from $\pi^\epsilon_0(X)$ to $\pi^{\epsilon'}_0(X)$ via $[x]_\epsilon \mapsto [x]_{\epsilon'}$. The collection $\{\mathrm{P}\Pi^\epsilon_0(X) := \pi^\epsilon_0(X)\}_{\epsilon \geqslant 0}$ together with the natural maps forms a persistent set, i.e., an object in $\mathbf{PSet}^{(\mathbb{R}_{\geqslant 0}, \leqslant)}$, denoted by $\mathrm{P}\Pi_0(X)$.

Fact 4.1. $\mathcal{V}_{\Bbbk}(\mathrm{P\Pi}_0(X)) \cong \mathrm{PH}_0(X;\Bbbk)$.

Proof. Fix an $\epsilon \geqslant 0$. Clearly, the boundary operator

$$\partial_1 : C_1(\mathrm{VR}_\epsilon(X);\Bbbk) = \mathcal{V}_{\Bbbk}(\{(x,x') \in X \times X : d_X(x,x') \leqslant \epsilon\}) \to C_0(\mathrm{VR}_\epsilon(X);\Bbbk) = \mathcal{V}_{\Bbbk}(X)$$

is given by $(x,x') \mapsto x - x'$. Thus, $\mathrm{Im}(\partial_1) = \mathcal{V}_{\Bbbk}\{x'-x : d_X(x,x') \leqslant \epsilon\}$ and

$$\begin{aligned}
\mathrm{PH}_0^\epsilon(X;\Bbbk) &= \mathrm{H}_0(\mathrm{VR}_\epsilon(X);\Bbbk)\\
&= \mathcal{V}_{\Bbbk}(X)/\mathcal{V}_{\Bbbk}\{x'-x : d_X(x,x') \leqslant \epsilon\}\\
&\cong \mathcal{V}_{\Bbbk}(X/\sim_0^\epsilon) = \mathcal{V}_{\Bbbk}(\mathrm{P\Pi}_0^\epsilon(X)).
\end{aligned}$$

In addition, the above isomorphisms form a natural ismorphism between $\mathcal{V}_{\Bbbk}(\mathrm{P\Pi}_0(X))$ and $\mathrm{PH}_0(X;\Bbbk)$. ☐

Proposition 4.2. *The following defines an ultrametric on X: for any $x, x' \in X$,*

$$\begin{aligned}
\mu_X^{(0)}(x,x') &:= \inf\{\epsilon : x \sim_0^\epsilon x'\}\\
&= \inf\{\epsilon : \exists\{x_0,\cdots,x_n\} \subset X \ s.t.\ x_0 = x, x_n = x', d_X(x_i,x_{i+1}) \leqslant \epsilon\}.
\end{aligned}$$

Proof. Clearly, $\mu_X^{(0)}(x,x) = 0$ and $\mu_X^{(0)}(x,x') = \mu_X^{(0)}(x',x)$ for all x and $x' \in X$. It remains to prove the strong triangle inequality. Given arbitrary $x,y,z \in X$, suppose $\epsilon_1 > \mu_X^{(0)}(x,y)$ and $\epsilon_2 > \mu_X^{(0)}(y,z)$. Let $x_0^1 \cdots x_{n_1}^1$ and $x_0^2 \cdots x_{n_2}^2$ be the sequences to realize $x \sim_{\epsilon_1} y$ and $y \sim_{\epsilon_2} z$, respectively. Then $x_0^1 \cdots x_{n_1}^1 x_{n_2}^2 \cdots x_0^2$ is such that $x_0^1 = x, x_0^2 = z$ and the distance between adjacent points in that sequence is no larger than $\max\{\epsilon_1,\epsilon_2\}$. By the minimality, $\mu_X^{(0)}(x,z) \leqslant \max\{\epsilon_1,\epsilon_2\}$. Letting $\epsilon_1 \searrow \mu_X^{(0)}(x,y)$ and $\epsilon_2 \searrow \mu_X^{(0)}(y,z)$, we obtain the strong triangle inequality. ☐

A stability theorem for $\mu_\bullet^{(0)}$ can be found in [CM10]. We include it here together with a proof for pedagogical reasons: our proof of Theorem 4.23 will exhibit a similar pattern.

Theorem 4.3 (Stability Theorem for $\mu_\bullet^{(0)}$)**.** *Given (X,d_X) and $(Y,d_Y) \in \mathcal{M}$,*

$$d_{\mathrm{GH}}\left(\left(X,\mu_X^{(0)}\right),\left(Y,\mu_Y^{(0)}\right)\right) \leqslant d_{\mathrm{GH}}(X,Y).$$

Proof. Let $R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ be an arbitrary tripod between X and Y. Given (x,y) and (x',y') in R, suppose $\epsilon > \mu_X^{(0)}(x,x')$ and $x_0 \cdots x_n$ is a sequence to realize ϵ. For each $0 \leqslant i \leqslant n-1$, there exists some $z_i \in Z$ such that $\phi_X(z_i) = x_i$. Let $y_i = \phi_Y(z_i)$. For $0 \leqslant i \leqslant n-1$,

$$\begin{aligned}
d_Y(y_i,y_{i+1}) &\leqslant d_X(x_i,x_{i+1}) + |d_Y(y_i,y_{i+1}) - d_X(x_i,x_{i+1})|\\
&\leqslant \epsilon + \mathrm{dis}(R),
\end{aligned}$$

where $\mathrm{dis}(R)$ denotes the distortion induced by d_X and d_Y. Thus, $\{y = y_0, y_1, \cdots, y_{n-1}, y_n = y'\}$ is an $(\epsilon + \mathrm{dis}(R))$-homotopy between y and y'. As $\epsilon \searrow \mu_X^{(0)}(x,x')$, it follows that $\mu_Y^{(0)}(y,y') \leqslant \mu_X^{(0)}(x,x') + \mathrm{dis}(R)$. Similarly, we can prove $\mu_X^{(0)}(x,x') \leqslant \mu_Y^{(0)}(y,y') + \mathrm{dis}(R)$. Therefore,

$$\begin{aligned}
d_{\mathrm{GH}}((X,\mu_X^{(0)}),(Y,\mu_Y^{(0)})) &= \tfrac{1}{2}\inf_{R \in \mathfrak{R}(X,Y)} \max_{(x,y),(x',y') \in R} |\mu_X^{(0)}(x,x') - \mu_Y^{(0)}(y,y')|\\
&\leqslant \tfrac{1}{2}\inf_{R \in \mathfrak{R}(X,Y)} \mathrm{dis}(R)\\
&= d_{\mathrm{GH}}(X,Y).
\end{aligned}$$

□

The following example is a case where $\mu_\bullet^{(0)}$ gives a better approximation of the Gromov-Hausdorff distance, compared to the bottelneck distance lower bound given by Theorem 3.10.

Example 4.4. Let $X_\epsilon := (\{0,1\}, d_\epsilon)$ be a metric space consisting of two points, where $d_\epsilon(0,1) = 1+\epsilon$. Since $\mu^{(0)}_{X_\epsilon} = d_\epsilon$, we have

$$d_{\mathrm{GH}}\left(\left(X_\epsilon, \mu^{(0)}_{X_\epsilon}\right), \left(X_0, \mu^{(0)}_{X_0}\right)\right) = d_{\mathrm{GH}}(X_\epsilon, X_0) = \tfrac{\epsilon}{2},$$

for all $\epsilon \geqslant 0$. On the other hand, it is not hard to verify that $\mathrm{dgm}_0(X_\epsilon) = \{(0, 1+\epsilon), (0, +\infty)\}$, and thus,

$$\tfrac{1}{2} d_{\mathrm{B}}(\mathrm{dgm}_0(X_\epsilon), \mathrm{dgm}_0(X_0)) = \min\left\{\tfrac{\epsilon}{2}, \tfrac{1+\epsilon}{4}\right\} \leqslant \tfrac{\epsilon}{2},$$

where the inequality is strict when $\epsilon > 1$.

4.2. **Discrete Fundamental Groups.** Let us fix a parameter $\epsilon > 0$, and adopt the construction of discrete fundamental groups from [Wil11]. An *ϵ-chain* in (X, d_X) is a finite sequence of points $\gamma = x_0 \cdots x_n$ such that $d_X(x_i, x_{i+1}) \leqslant \epsilon$ for $i = 0, \cdots, n-1$. When $x_0 = x_n$, we say that γ is an *ϵ-loop*. The integer n is called the *size* of γ, and written as $\mathrm{size}(\gamma)$. The *reversal* of γ is the ϵ-chain $\gamma^{-1} := x_n x_{n-1} \cdots x_0$. The terms *discrete chains* (resp. *discrete loops*) will refer to ϵ-chains (resp. ϵ-loops) for all $\epsilon > 0$. A *subdivision* of a discrete chain γ is a discrete chain which can be written as

$$\alpha' = (x_0^0 \cdots x_0^{m_0}) \cdots (x_i^0 \cdots x_i^{m_i}) \cdots (x_{n-1}^0 \cdots x_{n-1}^{m_{n-1}}),$$

where $x_i^0 = x_i, x_i^{m_i} = x_{i+1}$ and $x_i^{j_i} \in X$ for any i and $j_i = 1, \cdots, m_i - 1$. In this case, we denote $\alpha' \supset \alpha$ or $\alpha \subset \alpha'$.

A space X is *ϵ-connected* if any two points $x, y \in X$ can be joined by an ϵ-chain. If X is ϵ-connected for all $\epsilon > 0$, we say that X is *chain-connected*.

Lemma 4.5 (Lemma 2.1.1, [Wil11])**.** *Let X be a metric space.*

- *If X is connected, then X is chain-connected.*
- *If X is chain-connected and compact, then X is connected.*

In [Wil11], Wilkins defined a *basic move* on an ϵ-chain as the addition or removal of a point which is not an endpoint, so that the resulting path is still an ϵ-chain. In our paper, in order to avoid tedious arguments, basic moves also include the case when there are no changes.

Definition 4.6. Two chains α and β are *ϵ-homotopic*, denoted by $\alpha \sim_1^\epsilon \beta$, if there is a finite sequence of ϵ-chains

$$H = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k-1}, \gamma_k = \beta\}$$

such that each γ_i differs from γ_{i-1} by a basic move. We call H an *ϵ-homotopy* and denote the ϵ-homotopy class of an ϵ-chain α by $[\alpha]_\epsilon$. If an ϵ-loop $\gamma = x_0 x_1 \cdots x_n$ is ϵ-homotopic to the trivial loop $\{x_0\}$, we say that γ is *ϵ-null*.

The following reformulation (see [BCW14]) will be useful in the sequel. For a non-negative integer n, let $[n] := \{0, \cdots, n-1\}$. Note that an ϵ-chain in X can also be regarded as an ϵ-Lipschitz map $\gamma : ([n], \ell^\infty) \to (X, d_X)$ for some $n \in \mathbb{Z}_{\geqslant 0}$, i.e., $d_X(\gamma(i), \gamma(i+1)) \leqslant \epsilon$ for all $i \in [n-1]$. A *lazification* of an ϵ-path $\gamma : [n] \to X$ is an ϵ-path $\tilde{\gamma} : [m] \to X$ such that $m \geqslant n$ and $\tilde{\gamma} = \gamma \circ p$ where $p : [m] \to [n]$ is surjective and monotone.

Proposition 4.7. *Two chains α and β are ϵ-*homotopic *if and only if there exists a triple $(\tilde{\alpha}, \tilde{\beta}, \tilde{H})$ where*

- *$\tilde{\alpha}$ and $\tilde{\beta}$, of the same size n, are lazifications of α and β, respectively;*
- *$\tilde{H} : ([n] \times [m], \ell^\infty) \to (X, d_X)$ is an ϵ-Lipschitz map such that $\tilde{H}(\cdot, 0) = \alpha, \tilde{H}(\cdot, m) = \beta$, $\tilde{H}(0, t) = \alpha(0) = \beta(0)$ and $\tilde{H}(n, t) = \alpha(n) = \beta(n)$ for all $t \in [m]$.*

Proof. We prove the 'if' part by induction on the number of basic moves. For the base case, we suppose $\alpha = x_0 \cdots x_n$ and $\{\alpha = \gamma_0, \gamma_1 = \beta\}$ is an ϵ-homotopy. If there is no change between α and β, we take

$$\tilde{H}(\cdot, 0) = \gamma_0 \text{ and } \tilde{H}(\cdot, 1) = \gamma_1.$$

Otherwise, there is a removal or addition of a point from α to β. Define two maps $p_i : [n] \to [n-1]$ and $p^i : [n+1] \to [n]$ as

$$p_i(j) = \begin{cases} j, & \text{if } j \neq i \\ i-1, & \text{if } j = i \end{cases} \text{ and } p^i(j) = \begin{cases} j & \text{if } j < i, \\ i-1, & \text{if } j \geqslant i \end{cases}.$$

If γ_1 is obtained from γ_0 by removing some point x_i, we set $\tilde{\gamma_1} = \gamma_1 \circ p_i$ and

$$\tilde{H}(\cdot, 0) = \gamma_0 \text{ and } \tilde{H}(\cdot, 1) = \tilde{\gamma_1}.$$

If γ_1 is obtained from γ_0 by adding some point between x_{i-1} and x_i, we take $\tilde{\gamma_0} = \gamma_1 \circ p^i$

$$\tilde{H}(\cdot, 0) = \tilde{\gamma_0} \text{ and } \tilde{H}(\cdot, 1) = \gamma_1.$$

Now suppose the statement is true for any two ϵ-chains which differ by $k-1$ basic moves. Assume that $H = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k-1}, \gamma_k = \beta\}$ is an ϵ-homotopy between α and β. By the induction hypothesis, there exists a required triple $(\tilde{\gamma_0}, \tilde{\gamma_{k-1}}, \tilde{H}_1)$. Since γ_{k-1} and γ_k differ by one basic move, we need to discuss three cases as before. Notice that $\tilde{H}_1$ is essentially a matrix with entries from X. If there is no change, we append one more row γ_k to $\tilde{H}_1$ to obtain $\tilde{H}$. If γ_{k-1} and γ_k differ by the removal of the point $\gamma_{k-1}(i)$, we compose γ_k with some $p_i : [\text{size}(\gamma_{k-1})] \to [\text{size}(\gamma_k)]$ and append $\gamma_k \circ p_i$ as a new row to $\tilde{H}_1$. If they differ by the addition of the point $\gamma_k(i)$, we compose each row of $\tilde{H}_1$ with some $p^i : [\text{size}(\gamma_k)] \to [\text{size}(\gamma_{k-1})]$ and append one more row γ_k to $\tilde{H}_1$.

For the 'only if' part, it suffices to establish it for the case $m = 1$. Suppose $\beta = x_0' x_1' \cdots x_{n-1}' x_n'$, where $x_0' = x_0$ and $x_n' = x_n$. For each $i = 0, \cdots, n-1$, we have $d_X(x_i, x_{i+1}') \leqslant \epsilon$. Thus, x_i' can be inserted between x_{i-1} and x_i into α for each $i = 1, \cdots, n$, after which x_i can be removed for each $i = 1, \cdots, n$. Therefore, we can obtain β from α via $2n$ basic moves. □

The following fact will be used repeatedly.

Remark 4.8. Let $p_j : [n_j] \to [n]$ for $j = 1, 2$ be surjective monotone maps. There exists an integer N, and surjective monotone maps $q_j : [N] \to [n_j]$ for $j = 1, 2$ such that $p_1 \circ q_1 = p_2 \circ q_2$.

Fix a basepoint $x_0 \in X$ and $\epsilon > 0$.

Proposition 4.9. *The relation $\sim_1^\epsilon$ defines an equivalence relation on the set of discrete chains based at x_0, denoted by $\mathcal{L}(X, x_0)$.*

Proof. Clearly, $\sim_1^\epsilon$ is reflexive and symmetric. Let α, β and γ be any three chains such that $\alpha \sim_1^\epsilon \beta$ and $\beta \sim_1^\epsilon \gamma$. Then, there exists two sequences of ϵ-chains $H_1 = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k-1}, \gamma_k = \beta\}$ and $H_2 = \{\beta = \gamma_{k+1}, \gamma_{k+2}, \cdots, \gamma_{k+l-1}, \gamma_{k+l} = \gamma\}$ such that each

γ_i differs from γ_{i-1} by a basic move. Then $H = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k+l-1}, \gamma_{k+l} = \gamma\}$ is a sequence of ϵ-chains, where each adjacent two differ by a basic move. Thus, $\alpha \sim_1^\epsilon \gamma$. □

Let $\mathcal{L}^\epsilon(X, x_0)$ be the collection of all ϵ-loops based at x_0. Given two ϵ-loops $\gamma = x_0 x_1 \cdots x_{n-1} x_0$ and $\gamma' = x_0 y_1 \cdots y_{m-1} x_0$, define their *concatenation* by

$$\gamma * \gamma' = x_0 x_1 \cdots x_{n-1} x_0 y_1 \cdots y_{m-1} x_0.$$

The *birth time* of a discrete loop $\gamma = x_0 x_1 \cdots x_n$ is defined to be

$$\operatorname{birth}(\gamma) := \max_{0 \leqslant i \leqslant n-1} d_X(x_i, x_{i+1}).$$

The *death time* of γ is defined to be

$$\operatorname{death}(\gamma) := \inf\{\epsilon > 0 : \gamma \sim_1^\epsilon \{x_0\}\}.$$

When X is compact, it is clear that both $\operatorname{birth}(\gamma)$ and $\operatorname{death}(\gamma)$ are no larger than $\operatorname{diam}(X)$.

Remark 4.10. Because $\mathcal{L}^\epsilon(X, x_0)$ is a subset of $\mathcal{L}(X, x_0)$, it follows from Proposition 4.9 that $\sim_1^\epsilon$ defines an equivalence relation on $\mathcal{L}^\epsilon(X, x_0)$. By Lemma 2.1.2 of [Wil11], $\mathcal{L}^\epsilon(X, x_0)/\sim_1^\epsilon$ is a group under the operation of concatenation, where the identity element is $[x_0]_\epsilon$ and $[\gamma]_\epsilon^{-1} = [\gamma^{-1}]_\epsilon$.

Definition 4.11 (Discrete fundamental group)**.** The *discrete fundamental group at scale* ϵ of a metric space X, based at x_0, is

$$\pi_1^\epsilon(X, x_0) := \mathcal{L}^\epsilon(X, x_0)/\sim_1^\epsilon .$$

Proposition 4.12 (Lemma 2.2.9, [Wil11])**.** *For $\epsilon > 0$, suppose x_0 and x_1 are points in X such that there is an ϵ-chain λ from x_0 to x_1. Then*

$$\pi_1^\epsilon(X, x_0) \xrightarrow{\cong} \pi_1^\epsilon(X, x_1) \text{ via } [\gamma]_\epsilon \mapsto [\lambda^{-1} * \gamma * \lambda]_\epsilon.$$

Thus, if X is an ϵ-connected metric space, we have $\pi_1^\epsilon(X, x_0) \cong \pi_1^\epsilon(X, x_1)$ for any two points x_0 and x_1 in X. In this case, the discrete fundamental groups at scale ϵ are independent of choices of basepoints, up to isomorphism, so we can omit the basepoints and simply write $\pi_1^\epsilon(X)$.

Example 4.13 (Example 2.2.6, [Wil11])**.** Let $\mathbb{S}^1(r)$ be a circle of radius $r > 0$, equipped with the geodesic metric. When $r = 1$, we simply write $\mathbb{S}^1$ for $\mathbb{S}^1(1)$. Then

$$\pi_1^\epsilon(\mathbb{S}^1(r)) = \begin{cases} \mathbb{Z}, & \text{if } 0 < \epsilon < \frac{2\pi}{3} r, \\ 0, & \text{if } \epsilon \geqslant \frac{2\pi}{3} r. \end{cases}$$

In Theorem 4.15, we will see that discrete fundamental groups of a compact metric space are in fact isomorphic to fundamental groups of its successive Vietoris-Rips complexes.

Edge-path Groups of Vietoris-Rips Complexes. Given a simplicial complex K, an *edge* in K is an ordered pair $e = v_1 v_2$ for vertices of K, such that v_1 and v_2 are in the same simplex. An *edge path* in K is a sequence of vertices connected by edges. The concatenation of two edge paths can be defined in a natural way and called the *product* of two edge paths. If $e = v_1 v_2$ and $f = v_2 v_3$ are such that v_1, v_2 and v_3 are vertices of a simplex, then the product ef is *edge equivalent* to $v_1 v_3$. Two edge paths are *edge equivalent* if one can be obtained from another by a sequence of such elementary edge equivalences. Let v_0 be a vertex of K. When an edge starts and ends at the same vertex v_0, we call it an *edge loop at* v_0. We define

$\pi_1^{\mathrm{E}}(K, v_0)$ to be the set of edge equivalence classes of edge loops at v_0, called the *edge-path group* of K (see [ST67, p. 87]). Let $|K|$ denote the geometric realization of K.

Theorem 4.14 (Theorem 4 and 5, §4.4, [ST67])**.** *With the notations above, $\pi_1^{\mathrm{E}}(K, v_0)$ is a group, with identity v_0v_0, under the operation of product defined above. Furthermore,*

$$\pi_1^{\mathrm{E}}(K, v_0) \cong \pi_1(|K|, v_0).$$

It is an interesting fact that the following groups are isomorphic.

Theorem 4.15 (Isomorphisms)**.** *Let (X, x_0) be a pointed compact metric space and let $\epsilon \geqslant 0$. Then*

$$\pi_1^{\epsilon}(X, x_0) \cong \pi_1^{\mathrm{E}}(\mathrm{VR}_\epsilon(X), x_0) \cong \pi_1\left(|\mathrm{VR}_\epsilon(X)|, x_0\right).$$

Proof. The rightmost isomorphism follows directly from Theorem 4.14. The leftmost isomorphism is straightforward from the definitions. Indeed, if we write $\mathcal{L}^{\mathrm{E}}(\mathrm{VR}_\epsilon(X), x_0)$ as the set of loops at x_0 in $\mathrm{VR}_\epsilon(X)$, it is clear that $\mathcal{L}^{\epsilon}(X, x_0) = \mathcal{L}^{\mathrm{E}}(\mathrm{VR}_\epsilon(X, x_0))$ as sets. It remains to check that ϵ-homotopy equivalence is the same as the edge equivalence in $\mathcal{L}^{\mathrm{E}}(\mathrm{VR}_\epsilon(X, x_0))$, which can be done by proving that basic moves are equivalent to elementary edge equivalences. Indeed, for $x_1, x_3 \in X$ such that $d_1(x_1, x_3) \leqslant \epsilon$, $x_1x_2x_3 \sim_1^{\mathrm{E}} x_1x_3$ iff x_1, x_2 and x_3 form a 2-simplex in $\mathrm{VR}_\epsilon(X, x_0)$, iff $x_1x_2x_3 \sim_1^{\epsilon} x_1x_3$. □

Given $\epsilon' > \epsilon$, an ϵ-chain is also an ϵ'-chain and an ϵ-homotopy is also an ϵ'-homotopy. Thus, there is a natural group homomorphism

$$\Phi_{\epsilon,\epsilon'} : \pi_1^{\epsilon}(X, x_0) \to \pi_1^{\epsilon'}(X, x_0) \text{ with } [\alpha]_\epsilon \mapsto [\alpha]_{\epsilon'}.$$

The collection $\{\mathrm{P\Pi}_1^{\epsilon}(X, x_0) := \pi_1^{\epsilon}(X, x_0)\}_{\epsilon>0}$ together with the natural group homomorphisms $\{\Phi_{\epsilon,\epsilon'}\}_{\epsilon'\geqslant\epsilon>0}$ forms a persistent group, denoted by $\mathrm{P\Pi}_1^{\bullet}(X, x_0)$, or $\mathrm{P\Pi}_1(X, x_0)$ for simplicity.

Remark 4.16. The leftmost isomorphism in Theorem 4.15 was first establised in [Pla07, Page 599].

4.3. **Discretization Homomorphism.** Let $\gamma : [0, 1] \to X$ be a continuous path and $\epsilon > 0$. An *ϵ-chain along* γ is an ϵ-chain $x_0 \cdots x_n$ where there exists a partition $\{0 = t_0, \cdots, t_n = 1\}$ of $[0, 1]$ such that each $x_i = \gamma(t_i)$. A *strong ϵ-chain along* γ is an ϵ-chain along γ such that $\gamma([t_{i-1}, t_i]) \subset B(\gamma(t_{i-1}), \epsilon)$ for each i. When γ is a continuous loop, a (strong) ϵ-chain along γ is also called a *(strong) ϵ-loop along* γ.

The following lemma permits relating the fundamental group of a space to its discrete fundamental groups.

Lemma 4.17 (Lemma 3.1.7, [Wil11])**.** *Let X be a chain-connected metric space, and let $\epsilon > 0$ be given. Then the following map is a well-defined homomorphism (called the ϵ-discretization homomorphism)*

$$\Phi_\epsilon : \pi_1(X) \to \pi_1^{\epsilon}(X) \text{ with } [\gamma] \mapsto [\alpha]_\epsilon,$$

where α is a strong ϵ-loop along γ. If the ϵ-balls of X are path-connected (e.g. X is geodesic or locally simply connected), then Φ_ϵ is surjective.

Example 4.18. Let Y be any simply connected (thus, geodesic) compact space. By Lemma 4.17 and the fact $\pi_1(Y) = 0$, we have

$$\pi_1^{\epsilon}(Y) = 0, \forall \epsilon > 0.$$

The well-definedness of Φ_ϵ indicates that any two ϵ-discretizations of a continuous loop are ϵ-homotopic. It follows that $\Phi_{\epsilon,\epsilon'} \circ \Phi_\epsilon = \Phi_{\epsilon'}$ for all $\epsilon \leqslant \epsilon'$. By the universal property of limit, there is a natural group homomorphism

$$\Phi : \pi_1(X) \to \lim \mathrm{P}\Pi_1(X) \text{ with } [\gamma] \mapsto \lim[\alpha]_\epsilon.$$

Vigolo proved in [Vig18, Theorem 3.2] that for u.l.p.c. and u.s.l.s.c. metric spaces the discretization homomorphism is in fact an isomorphism, albeit using a slightly different definition for ϵ-homotopy. It turns out a similar result and proof can be applied to our case as well.

Theorem 4.19 (Discretization Theorem)**.** *When a metric space (X, d_X) is u.l.p.c. and u.s.l.s.c., the ϵ-discretization homomorphism Φ_ϵ is an isomorphism for ϵ small enough. In addition, the discretization homomorphism $\Phi : \pi_1(X, x_0) \to \lim \mathrm{P}\Pi_1(X, x_0)$ is an isomorphism.*

Proof. We first prove that Φ_ϵ is injective for ϵ small enough. Since X is u.s.l.s.c., there exists $\delta' > 0$ such that a loop in $B(x, \delta')$ is null-homotopic for any $x \in X$. Because X is u.l.p.c., there exists $\delta < \delta'/2$ so that any two points in $B(x, \delta)$ can be connected by a path with image completely contained in $B(x, \delta'/2)$, for any $x \in X$. Fix a parameter $0 < \epsilon < \delta$ and let γ be a continous loop based at x_0 whose ϵ-discretization $\alpha = x_0 \cdots x_n$ is trivial. Recall that α is a strong ϵ-path along γ, i.e., there exists a partition $\{t_0, \cdots, t_n\}$ of $[0, 1]$ such that each $x_i = \gamma(t_i)$ and $\gamma([t_i, t_{i+1}]) \subset B(\gamma(t_i), \epsilon)$ for each i. We want to show γ is null-homotopic.

Suppose $\alpha \sim_1^\epsilon x_0$ via an ϵ-homotopy $H : [n] \times [m] \to X$ with $H(\cdot, 0) = \alpha$ and $H(\cdot, m) = x_0$. Let $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m$. Since $d_X(H(i, j), H(i+1, j)) \leqslant \epsilon < \delta$, there exists a path $\alpha_{i,j}$ joining them with image in $B(H(i, j), \delta'/2)$. Similarly, for $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m-1$, there exists a path $\beta_{i,j}$ joining $H(i, j)$ and $H(i, j+1)$ with image in $B(H(i, j), \delta'/2)$. It follows that the concatenations $\beta_{i,j}\alpha_{i,j+1}$ and $\alpha_{i,j}\beta_{i+1,j}$ are both contained in $B(H(i, j), \delta')$. Since $\alpha_{i,j}\beta_{i+1,j}\alpha_{i,j+1}^{-1}\beta_{i,j}^{-1}$ is a loop in $B(H(i, j), \delta')$, it is null-homotopic. For $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m-1$, let $\xi_{i,j} := \alpha_{0,0} \cdots \alpha_{i-1,0}\beta_{0,0} \cdots \beta_{i,j-1}$, a loop joining x_0 to $H(i, j)$, and then define $\eta_{i,j} := \xi_{i,j}\alpha_{i,j}\beta_{i+1,j}\alpha_{i,j+1}^{-1}\beta_{i,j}^{-1}\xi_{i,j}^{-1}$, which is a null-homotopic loop at x_0 based on construction. Notice that

$$\gamma_i := \eta_{i,0} \cdots \eta_{i,m-1} \sim (\alpha_{0,0} \cdots \alpha_{i,0})(\beta_{i+1,0} \cdots \beta_{i+1,m-1})\alpha_{i,m}^{-1}(\beta_{i,m-1}^{-1} \cdots \beta_{i,0}^{-1})(\alpha_{0,0} \cdots \alpha_{i,0})^{-1}$$

and

$$\gamma_{n-1} \cdots \gamma_0 \sim (\alpha_{0,0} \cdots \alpha_{n-1,0})(\beta_{n,0} \cdots \beta_{n,m-1})(\alpha_{n,m}^{-1} \cdots \alpha_{0,m}^{-1})(\beta_{0,m-1}^{-1} \cdots \beta_{0,0}^{-1}) \sim \alpha_{0,0} \cdots \alpha_{n-1,0}.$$

Since $\gamma([t_i, t_{i+1}]) * \alpha_{i,0}^{-1} \subset B(\gamma(t_i), \epsilon) \subset B(\gamma(t_i), \delta')$, $\gamma([t_i, t_{i+1}])$ must be a null-homotopic loop. Therefore, $\gamma \sim \alpha_{0,0} \cdots \alpha_{n-1,0} \sim x_0$. It follows that Φ_ϵ is injective for all $0 < \epsilon < \delta$, and Φ is injective as well.

Because X is u.l.p.c., its ϵ-balls are path connected. By Lemma 4.17, Φ_ϵ is surjective, and thus an isomorphism. It remains to show Φ is surjective. A generic element of $\lim \pi_1^\theta(X)$ can be represented by a family $([\alpha_\theta]_\theta)_{\theta>0}$, where α_θ is an θ-loop based at x_0 such that for every $\theta < \theta'$ we have $\alpha_\theta \sim_1^{\theta'} \alpha_{\theta'}$. Take a real number ϵ such that $0 < \epsilon < \delta$. The surjectivity of Φ_ϵ shows us that there is a continous loop γ_ϵ such that α_ϵ is ϵ-homotopic to a strong ϵ-path along γ_ϵ. Since $[\alpha_\epsilon]_{\epsilon'} = [\alpha_{\epsilon'}]_{\epsilon'}$ for $0 < \epsilon < \epsilon' < \delta$, the injectivity of $\Phi_{\epsilon'}$ implies that γ_ϵ and $\gamma_{\epsilon'}$ are homotopic. When $\theta \geqslant \epsilon$, we always have $[\alpha_\theta]_\theta = [\alpha_\epsilon]_\theta$. Thus, for $([\alpha_\theta]_\theta)_{\theta>0}$, we have found a continuous loop γ_ϵ such that

$$\Phi([\gamma_\epsilon]) = ([\gamma_\epsilon]_\theta)_{\theta>0} = ([\alpha_\theta]_\theta)_{\theta>0},$$

where $[\gamma_\epsilon]_\theta$ denotes the θ-homotopy class of a strong θ-path along γ_ϵ. □

4.4. **Generalized Subdendrograms and a Pseudo-metric on $\mathcal{L}(X, x_0)$.** By analogy with the ultrametric $\mu^{(0)}$ discussed in §4.1, we introduce a pseudo-ultrametric on the set of all discrete loops on a given metric space. This will allow us to view the discrete loop space of a metric space as a metric space in itself.

Proposition 4.20. *Given a pointed metric space (X, x_0), the following defines a pseudo-ultrametric on $\mathcal{L}(X, x_0)$: for any $\gamma, \gamma' \in \mathcal{L}(X, x_0)$,*

$$\mu_X^{(1)}(\gamma, \gamma') := \inf\{\epsilon > 0 : \gamma \sim_1^\epsilon \gamma'\}.$$

Proof. Clearly, $\mu_X^{(1)}(\gamma, \gamma') = \mu_X^{(1)}(\gamma', \gamma)$ for all $\gamma, \gamma' \in \mathcal{L}(X, x_0)$. It remains to prove the strong triangle inequality. Given arbitrary $\alpha, \beta, \gamma \in \mathcal{L}(X, x_0)$, let $\epsilon_1 = \mu_X^{(1)}(\alpha, \beta)$ and $\epsilon_2 = \mu_X^{(1)}(\beta, \gamma)$. For each $\delta_1 > \epsilon_1$ and $\delta_2 > \epsilon_2$, there exist a sequence of δ_1-paths $H_1 = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k-1}, \gamma_k = \beta\}$ and a sequence of δ_2-paths $H_2 = \{\beta = \gamma_{k+1}, \gamma_{k+2}, \cdots, \gamma_{k+l-1}, \gamma_{k+l} = \gamma\}$ such that each γ_i differs from γ_{i-1} by a basic move. Then $H = \{\alpha = \gamma_0, \gamma_1, \cdots, \gamma_{k+l-1}, \gamma_{k+l} = \gamma\}$ is a sequence of $\max\{\delta_1, \delta_2\}$-paths, where each adjacent two differ by a basic move. Then α and γ are $\max\{\delta_1, \delta_2\}$-homotopic via H. By the minimality, $\mu_X^{(1)}(\alpha, \gamma) \leqslant \max\{\delta_1, \delta_2\}$. Letting $\delta_i \searrow \epsilon_i$ $(i = 1, 2)$, we obtain $\mu_X^{(1)}(\alpha, \gamma) \leqslant \max\{\epsilon_1, \epsilon_2\}$. □

Remark 4.21. For any $\gamma \in \mathcal{L}(X, x_0)$, note that

$$\mathrm{birth}(\gamma) = \mu_X^{(1)}(\gamma, \gamma) \text{ and } \mathrm{death}(\gamma) = \mu_X^{(1)}(\gamma, \{x_0\}).$$

Suppose α and β are two discrete loops in X. Let $\epsilon = \max\{\mathrm{birth}(\alpha), \mathrm{birth}(\beta)\}$. Then, for any $\gamma \in \mathcal{L}^\epsilon(X, x_0)$, we have

$$\mu_X^{(1)}(\alpha * \gamma, \beta * \gamma) = \mu_X^{(1)}(\alpha, \beta).$$

This is because when $\delta \geqslant \epsilon$, there is a δ-homotopy $H = \{\alpha, \cdots, \beta\}$ iff there is a δ-homotopy $H * \gamma := \{\alpha * \gamma, \cdots, \beta * \gamma\}$.

Stability for $\mu_\bullet^{(1)}$. Let (X, x_0) and (Y, y_0) be in $\mathcal{M}^{\mathrm{pt}}$. Let $R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ be a pointed tripod between X and Y. Note that ϕ_X and ϕ_Y induce surjective maps

$$\mathcal{L}(Z, z_0) \twoheadrightarrow \mathcal{L}(X, x_0) \text{ and } \mathcal{L}(Z, z_0) \twoheadrightarrow \mathcal{L}(Y, y_0)$$

respectively, still denoted by ϕ_X and ϕ_Y. Thus, we have a tripod between $\left(\mathcal{L}(X, x_0), \mu_X^{(1)}\right)$ and $\left(\mathcal{L}(Y, y_0), \mu_Y^{(1)}\right)$:

$$R_{\mathcal{L}} := \mathcal{L}(X, x_0) \xleftarrow{\phi_X} \mathcal{L}(Z, z_0) \xrightarrow{\phi_Y} \mathcal{L}(Y, y_0).$$

Lemma 4.22. *Let (X, x_0) and (Y, y_0) be in $\mathcal{M}^{\mathrm{pt}}$. Then each $R \in \mathfrak{R}^{\mathrm{pt}}((X, x_0), (Y, y_0))$ induces a tripod $R_{\mathcal{L}} \in \mathfrak{R}(\mathcal{L}(X, x_0), \mathcal{L}(Y, y_0))$ with $\mathrm{dis}(R_{\mathcal{L}}) \leqslant \mathrm{dis}(R)$. In particular, if α_X is an ϵ-chain in X and $(\alpha_X, \alpha_Y) \in R_{\mathcal{L}}$ (see Page 11), then α_Y is an $(\epsilon + \mathrm{dis}(R))$-chain in Y.*

Proof. Let $R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ be a pointed tripod between X and Y. If $\alpha = x_0 \cdots x_n$ is an ϵ-loop in X, then there exists a discrete loop $\gamma = z_0 \cdots z_n$ in Z such that $\alpha = \phi_X(\gamma)$.

Note that $\phi_Y(\gamma)$ is an $(\epsilon + \mathrm{dis}(R))$-loop in Y, because for $0 \leqslant i \leqslant n-1$,

$$\begin{aligned} d_Y(\phi_Y(z_i), \phi_Y(z_{i+1})) &\leqslant d_X(\phi_X(z_i), \phi_X(z_{i+1})) + |d_Y(\phi_Y(z_i), \phi_Y(z_{i+1})) - d_X(\phi_X(z_i), \phi_X(z_{i+1}))| \\ &\leqslant \epsilon + \mathrm{dis}(R). \end{aligned}$$

Let $(\alpha, \beta), (\alpha', \beta') \in R_{\mathcal{L}}$ and $\delta > \mu_X^{(1)}(\alpha, \beta)$. Then there is a finite sequence of δ-loops in X

$$H_X = \{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_{k-1}, \alpha_k = \alpha'\}$$

such that each α_j differs from α_{j-1} by a basic move. For each α_j, let $\beta_j = \phi_Y(\gamma_j)$ where γ_j is such that $\alpha_j = \phi_X(\gamma_j)$. Then each β_j differs from β_{j-1} by a basic move. In particular,

$$H_Y = \{\beta = \beta_0, \beta_1, \cdots, \beta_{k-1}, \beta_k = \beta'\}$$

is a $(\delta + \mathrm{dis}(R))$-homotopy between β and β'. Therefore, $\mu_Y^{(1)}(\alpha', \beta') \leqslant \delta + \mathrm{dis}(R)$. Letting $\delta \searrow \mu_X^{(1)}(\alpha, \beta)$, we obtain $\mu_Y^{(1)}(\alpha', \beta') \leqslant \mu_X^{(1)}(\alpha, \beta) + \mathrm{dis}(R)$. Similarly, $\mu_X^{(1)}(\alpha, \beta) \leqslant \mu_Y^{(1)}(\alpha', \beta') + \mathrm{dis}(R)$. This is true for all $(\alpha, \beta), (\alpha', \beta') \in R_{\mathcal{L}}$, implying that $\mathrm{dis}(R_{\mathcal{L}}) \leqslant \mathrm{dis}(R)$. ☐

Theorem 4.23 (Stability Theorem for $\mu_\bullet^{(1)}$)**.** *Given (X, x_0) and (Y, y_0) in $\mathcal{M}^{\mathrm{pt}}$,*

$$d_{\mathrm{GH}}(\mathcal{L}(X, x_0), \mathcal{L}(Y, y_0)) \leqslant d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

Proof. By Lemma 4.22,

$$\begin{aligned} d_{\mathrm{GH}}(\mathcal{L}(X, x_0), \mathcal{L}(Y, y_0)) &= \tfrac{1}{2} \inf\{\mathrm{dis}(R_{\mathcal{L}}) : R_{\mathcal{L}} \in \mathfrak{R}(\mathcal{L}(X, x_0), \mathcal{L}(Y, y_0))\} \\ &\leqslant \tfrac{1}{2} \inf\{\mathrm{dis}(R_{\mathcal{L}}) : R_{\mathcal{L}} \text{ induced by } R, R \in \mathfrak{R}(X, Y)\} \\ &\leqslant \tfrac{1}{2} \inf\{\mathrm{dis}(R) : R \in \mathfrak{R}(X, Y)\} \\ &= d_{\mathrm{GH}}(X, Y). \end{aligned}$$

☐

Corollary 4.24. *Given (X, d_X) and $(Y, d_Y) \in \mathcal{M}$,*

$$\inf_{R \in \mathfrak{R}(X,Y)} \sup_{(x_0, y_0) \in R} d_{\mathrm{GH}}(\mathcal{L}(X, x_0), \mathcal{L}(Y, y_0)) \leqslant d_{\mathrm{GH}}(X, Y).$$

Critical Values. Let X be a chain-connected metric space. A *non-critical interval of* X is a non-empty open interval $I \subset \mathbb{R}_+$, such that for $\epsilon, \epsilon' \in I$ with $\epsilon < \epsilon'$, the map $\Phi_{\epsilon,\epsilon'}$ is bijective. We call a positive number ϵ a *critical value of* X if it is not contained in any non-critical interval. We denote the subset of $\mathbb{R}_+$ consisting of all critical values of X by $\mathrm{Cr}(X)$, and call this the *critical spectrum of* X (see [Wil11]).

Theorem 4.25 (Theorem 3.1.11, [Wil11])**.** *Let X be a compact geodesic space. Then* $\mathrm{Cr}(X)$ *is discrete and bounded above in* $\mathbb{R}_+$.

Remark 4.26. For a finite metric space X, it is straightforward to check that $\mathrm{Cr}(X)$ is discrete and bounded above in $\mathbb{R}_+$.

Given $(X, x_0) \in \mathcal{M}^{\mathrm{pt}}$ and $\epsilon > 0$, $\pi_1^\epsilon(X, x_0) = \mathcal{L}^\epsilon(X, x_0)/\sim_1^\epsilon$ forms a partition of $\mathcal{L}^\epsilon(X, x_0) \subset \mathcal{L}(X, x_0)$, i.e., a subpartition of $\mathcal{L}(X, x_0)$.

Lemma 4.27. *Let X be either a compact geodesic space or a finite metric space. The map $\epsilon \mapsto \pi_1^\epsilon(X, x_0)$ induces a generalized subdendrogram over $\mathcal{L}(X, x_0)$, denoted by $\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}$.*

Proof. We want to check that conditions (1'), (2) and (3) in Definition 3.16 are satisfied. For $\epsilon \leqslant \epsilon'$, $\mathcal{L}^{\epsilon}(X, x_0) \subset \mathcal{L}^{\epsilon'}(X, x_0)$ and $\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}(\epsilon)$ refines $\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}(\epsilon')|_{\mathcal{L}^{\epsilon}(X,x_0)}$, i.e., condition (1') is satisfied. For condition (3), we notice that whenever $\epsilon \geqslant \mathrm{diam}(X)$, $\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}(\epsilon)$ is the single block partition. The semi-continuity, i.e., condition (2): for all r there exists $\epsilon > 0$ s.t. $\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}(r) = \theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}(t)$ for all $t \in [r, r+\epsilon]$, follows from Theorem 4.25 and Remark 4.26. □

Examples of generalized subdendrograms arising from discrete fundamental groups will be presented in §7. It is not hard to see that $\mu^{(1)}_X = \mu^{\mathrm{s}}_{\theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}}$, so we can apply Proposition 3.17 to conclude:

Corollary 4.28. *Given* (X, d_X) *and* $(Y, d_Y) \in \mathcal{M}$,

$$\inf_{R \in \mathfrak{R}(X,Y)} \sup_{(x_0,y_0) \in R} d_{\mathrm{I}}\left(\mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}, \mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_{\mathcal{L}(Y,y_0)}\right) \leqslant d_{\mathrm{GH}}(X, Y).$$

In general, the computation of Gromov-Hausdorff distance is NP-hard [Sch17], while in the left-hand-side $d_{\mathrm{I}}\left(\mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_{\mathcal{L}(X,x_0)}, \mathcal{V}_{\Bbbk} \circ \theta^{\mathrm{s}}_{\mathcal{L}(Y,y_0)}\right)$, as the interleaving distance between 1-dimensional persistence modules, is computable in polynomial time.

5. Persistent Homotopy Groups

We assume $T = \mathbb{R}_+$ in this section unless otherwise specified. In §3.4, we have defined persistent groups as objects in $\mathbf{PGrp}^{(T,\leqslant)}$, as well as homomorphisms and homomorphisms of degree δ between persistent groups. In this section, we study persistent homotopy groups, which are persistent groups given by homotopy groups of some spaces.

Persistent Groups. Recall from §3.4 that the set of homorphisms from the persistent group $\mathbb{G}$ to the persistent group $\mathbb{H}$ is denoted by $\mathrm{Hom}(\mathbb{G}, \mathbb{H})$, and the collection of homorphisms of degree δ is written as $\mathrm{Hom}^{\delta}(\mathbb{G}, \mathbb{H})$. By *isomorphisms* between persistent groups, we will mean isomorphisms in the category $\mathbf{PGrp}^{(T,\leqslant)}$, denoted by $\cong$. We denote by $\langle 0 \rangle$ the trivial group containing exactly one element 0, the addition identity element. The trivial persistent group is given by

$$\mathbb{0} := \langle 0 \rangle [T].$$

For a group G and an interval $I \subset T$, the interval generalized persistence module $G[I]$ (see Page 13) is also called the *interval persistent group.*

From the definitions of product and coproduct given in §3.1, we immediately obtain the following proposition.

Proposition 5.1 (Product and coproduct)**.** *In* $\mathbf{PGrp}^{(T,\leqslant)}$, *for any two objects* $\mathbb{G}$ *and* $\mathbb{H}$,

- *the product is given by component-wise direct products and denoted by* $\mathbb{G} \times \mathbb{H}$*; and*
- *the coproduct is given by component-wise free products and denoted by* $\mathbb{G} * \mathbb{H}$.

5.1. Persistent Homotopy Groups.

Given a pointed metric space (X, x_0), there are different approaches to construct persistent groups using homotopy information from the metric space, such as the persistent group $\mathrm{P}\Pi_1(X, x_0)$ constructed on Page 22.

Definition 5.2 (Persistent fundamental group)**.** Given a pointed metric space (X, x_0), the persistent group $\mathrm{P}\Pi_1(X, x_0)$, given by $\{\pi_1^{\epsilon}(X, x_0)\}_{\epsilon > 0}$ together with the natural group homomorphisms $\{\Phi_{\epsilon,\epsilon'}\}_{\epsilon' \geqslant \epsilon > 0}$, is called the *persistent fundamental group* of X.

Let us examine some properties of persistent fundamental groups here.

Proposition 5.3 (Induced Homomorphisms)**.** *Let X and Y be chain-connected metric spaces, and let $\varphi : (X, x_0) \to (Y, y_0)$ be any pointed set map.*

(1) For any $\delta \geqslant \mathrm{dis}(\varphi)$ and $\epsilon > 0$, the map
$$\varphi_\epsilon^{\epsilon+\delta} : \pi_1^\epsilon(X) \to \pi_1^{\epsilon+\delta}(Y) \text{ with } [\alpha]_\epsilon \mapsto [\varphi(\alpha)]_{\epsilon+\delta}$$
is a well-defined group homomorphism, and $(\varphi_\epsilon^{\epsilon+\delta})_{\epsilon>0} \in \mathrm{Hom}^\delta(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y))$.

(2) If φ is 1-Lipschitz, then for each $\epsilon > 0$, the map
$$\varphi_\epsilon : \pi_1^\epsilon(X) \to \pi_1^\epsilon(Y) \text{ with } [\alpha]_\epsilon \mapsto [\varphi(\alpha)]_\epsilon$$
is a well-defined group homomorphism, and $(\varphi_\epsilon)_{\epsilon>0} \in \mathrm{Hom}(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y))$.

Proof. Given an ϵ-loop $\alpha = x_0 \cdots x_n$ in X, $\varphi(\alpha) = \varphi(x_0) \cdots \varphi(x_n)$ is an $(\epsilon + \delta)$-loop in Y, because for $0 \leqslant i \leqslant n-1$,
$$d_Y(\varphi(x_i), \varphi(x_{i+1})) \leqslant d_X(x_i, x_{i+1}) + |d_Y(\varphi(x_i), \varphi(x_{i+1})) - d_X(x_i, x_{i+1})| \leqslant \epsilon + \delta.$$
Since φ preserves basic moves, given an ϵ-homotopy $H = \{\alpha = \alpha_0, \cdots, \alpha_k = \beta\}$, $\varphi(H) := \{\varphi(\alpha) = \varphi(\alpha_0), \cdots, \varphi(\alpha_k) = \varphi(\beta)\}$ is an $(\epsilon + \delta)$-homotopy in Y. Thus, $\varphi_\epsilon^{\epsilon+\delta}$ is well-defined. In addition, $\varphi_\epsilon^{\epsilon+\delta}$ is a group homomorphism, because for any $\alpha, \beta \in \mathcal{L}(X, x_0)$,
$$\varphi_\epsilon^{\epsilon+\delta}([\alpha*\beta]_\epsilon) = [\varphi(\alpha*\beta)]_{\epsilon+\delta} = [\varphi(\alpha)*\varphi(\beta)]_{\epsilon+\delta} = [\varphi(\alpha)]_{\epsilon+\delta}[\varphi(\beta)]_{\epsilon+\delta} = \varphi_\epsilon^{\epsilon+\delta}([\alpha]_\epsilon)\varphi_\epsilon^{\epsilon+\delta}([\beta]_{\epsilon+\delta}).$$
Here $*$ denote the concatenation of discrete chains.

The statement $(\varphi_\epsilon^{\epsilon+\delta}) \in \mathrm{Hom}^\delta(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y))$ follows from the following commutative diagram (for all $\epsilon' > \epsilon$):
$$\begin{array}{ccc} [\alpha]_\epsilon & \longmapsto & [\varphi(\alpha)]_{\epsilon+\delta} \\ \downarrow & & \downarrow \\ [\alpha]_{\epsilon'} & \longmapsto & [\varphi(\alpha)]_{\epsilon'+\delta}. \end{array}$$

Similar arguments can be applied to prove Part (2), so we omit them. ☐

5.1.1. *Persistent* K*-homotopy Groups and Persistent* VR*-homotopy Groups.* For $n \in \mathbb{Z}_{\geqslant 1}$, assigning the n-dimensional homotopy group to a pointed topological space forms a functor from the category of pointed spaces to the category of groups, called the *homotopy group functor* and denoted by $\pi_n : \mathbf{Top}^* \to \mathbf{Grp}$. In particular, when $n = 1$, we call it the *fundamental group functor*. In analogy to persistent homology theory, one could apply the homotopy group functor to filtrations of a given metric space to obtain a persistent group.

Kuratowski Filtration and Persistent K**-homotopy Groups.** Let (X, d_X) be a bounded metric space. We recall from [Bor67] a method to embed X isometrically as a subset of some Banach space, which will eventually allow us to enlarge X. Let $K(X) := (L^\infty(X), \|\cdot\|_\infty)$, where $L^\infty(X)$ is the space of bounded functions $f : X \to \mathbb{R}$ and $\|f\|_\infty := \sup_{x\in X} |f(x)|$. For a bounded metric space (X, d_X), the *Kuratowski embedding* is given by
$$\begin{aligned} k_X : X &\to L^\infty(X), \\ x &\mapsto d_X(x, \cdot). \end{aligned}$$
By the Kuratowski–Wojdysławski Theorem (see [Bor67, Theorem III.8.1]), the Kuratowski embedding k_X is distance-preserving.

Definition 5.4 (ϵ-thickening)**.** The ϵ-*thickening* of a compact metric space (X, d_X) is the closed (or open) ϵ-neighborhood of $k_X(X)$ in $K(X)$, henceforth denoted by X^ϵ (or $X^{<\epsilon}$).

Let (X, x_0) be a pointed compact metric space. For any $0 \leqslant \epsilon < \epsilon'$, there is a natural embedding $i^X_{\epsilon,\epsilon'} : X^\epsilon \to X^{\epsilon'}$, which forms a filtration in **Top**, called the *Kuratowski filtration* and denoted by $\mathbb{X}$. When there is no danger of confusion, we simply write i^X for $i^X_{\epsilon,\epsilon'}$. Under the map k_X, x_0 can be viewed as a point in X^ϵ for each $\epsilon > 0$, which induces a functor $(\mathbb{X}, x_0) : (\mathbb{R}_{\geqslant 0}, \leqslant) \to \mathbf{Top}^*$ such that $(\mathbb{X}, x_0)(\epsilon) = (X^\epsilon, x_0)$ and $(\mathbb{X}, x_0)(\epsilon \leqslant \epsilon') = i^X_{\epsilon,\epsilon'}$.

Definition 5.5 (Persistent K-homotopy group[1])**.** Let (X, x_0) be a pointed compact metric space. By $(\mathbb{X}, x_0)$, composing the homotopy group functor π_n with $(\mathbb{X}, x_0)$ induces a persistent group

$$\mathrm{P\Pi}^{\mathrm{K}}_n(X, x_0) := \{\pi_n(X^\epsilon, x_0)\}_{\epsilon \geqslant 0} ,$$

together with the induced homomorphisms on homotopy groups. We call it the *n-th persistent* K*-homotopy group.*

Remark 5.6. Composing the homology functor H_n with $\mathbb{X}$ results into the *persistent* K*-homology group* $\mathrm{PH}^{\mathrm{K}}_n(X, x_0)$.

Similarly, we can apply homotopy group functors to the geometric realizations of Vietoris-Rips complexes.

Definition 5.7 (Persistent VR-homotopy group)**.** Let (X, x_0) be a pointed compact metric space. For each $n \in \mathbb{Z}_{\geqslant 1}$, we have the persistent group

$$\mathrm{P\Pi}^{\mathrm{VR}}_n(X, x_0) := \{\pi_n\left(|\mathrm{VR}_\epsilon(X)| , x_0\right)\}_{\epsilon \geqslant 0} ,$$

together with the induced homomorphisms. We call it the *n-th persistent* VR*-homotopy group.*

As Vietoris-Rips complexes are simplicial complexes, one can also apply the edge-path group (see Page 22) functor to obtain a persistent group

$$\mathrm{P\Pi}^{\mathrm{E}}_1(X, x_0) := \left\{\pi^{\mathrm{E}}_1(\mathrm{VR}_\epsilon(X), x_0)\right\}_{\epsilon \geqslant 0} ,$$

together with the induced homomorphisms edge-path groups. We have seen in Theorem 4.15 that

$$\mathrm{P\Pi}^{\mathrm{VR}}_1(X, x_0) \cong \mathrm{P\Pi}^{\mathrm{E}}_1(X, x_0) \cong \mathrm{P\Pi}_1(X, x_0).$$

It will be seen in Theorem 5.12 that Definition $\mathrm{P\Pi}^{\mathrm{K}}_1(X, x_0)$ is isomorphic to the others as well.

Remark 5.8. One advantage of persistent K-homotopy groups and persistent VR-homotopy groups is that they can be defined for n-dimensional homotopy groups. However, it does not appear to be trivial to generalize the idea of persistent fundamental group or $\mathrm{P\Pi}^{\mathrm{E}}_1(X, x_0)$ to higher dimensions.

Theorem 5.9 (Theorem 3.1, [LMO20])**.** *Let $X \in \mathcal{M}$ and $\epsilon \geqslant 0$. Then,*

$$\mathbb{X}^{<\bullet} \cong |\mathrm{VR}_{<2\bullet}(X)| ,$$

as functors from $(\mathbb{R}_{\geqslant 0}, \leqslant)$ to the homotopy category of **Top***.*

[1] K is for Kuratowski.

Remark 5.10. By a very similar argument with Theorem 5.9, one can check that if X is a finite metric space, then

$$\mathbb{X}^{\leqslant \bullet} \cong \left|\mathrm{VR}_{\leqslant 2\bullet}(X)\right|.$$

If X is not finite, it does not appear to be a trivial question whether the above isomorphism still holds. But we always have that

$$d_{\mathrm{HI}}\left(\mathbb{X}^{\leqslant \bullet}, \left|\mathrm{VR}_{\leqslant 2\bullet}(X)\right|\right) = 0,$$

by approximating X with a sequence of finite metric spaces under d_{GH} and applying the triangle inequality, the stability of Kuratowski filtration and the Vietoris-Rips filtration. Indeed, let $\epsilon > 0$ be arbitrarily small, and let X_ϵ be a finite metric space such that $d_{\mathrm{GH}}(X, X_\epsilon) \leqslant \epsilon$. Then

$$\begin{aligned} d_{\mathrm{HI}}\left(\mathbb{X}^{\leqslant \bullet}, \left|\mathrm{VR}_{\leqslant 2\bullet}(X)\right|\right) &\leqslant d_{\mathrm{HI}}\left(\mathbb{X}^{\leqslant \bullet}, \mathbb{X}_\epsilon^{\leqslant \bullet}\right) + d_{\mathrm{HI}}\left(\mathbb{X}_\epsilon^{\leqslant \bullet}, \left|\mathrm{VR}_{\leqslant 2\bullet}(X_\epsilon)\right|\right) \\ &\quad + d_{\mathrm{HI}}\left(\left|\mathrm{VR}_{\leqslant 2\bullet}(X_\epsilon)\right|, \left|\mathrm{VR}_{\leqslant 2\bullet}(X)\right|\right) \\ &\leqslant \epsilon + 0 + \epsilon = 2\epsilon. \end{aligned}$$

Corollary 5.11. *Let (X, x_0) be a pointed compact metric space. Then,*

$$\mathrm{P}\Pi_n^{\mathrm{K},\bullet}(X, x_0) \cong \mathrm{P}\Pi_n^{\mathrm{VR},2\bullet}(X, x_0) \text{ and } \mathrm{PH}_n^{\mathrm{K},\bullet}(X, x_0) \cong \mathrm{PH}_n^{\mathrm{VR},2\bullet}(X, x_0),$$

which are true for the open version and for the closed version with X a finite metric space.

Now we are ready to prove the following isomorphism theorem of persistent fundamental groups.

Theorem 5.12 (Isomorphisms of persistent fundamental groups)**.** *Given a pointed compact metric space (X, x_0), we have*

$$\mathrm{P}\Pi_1^{\mathrm{K},\bullet}(X, x_0) \cong \mathrm{P}\Pi_1^{\mathrm{VR},2\bullet}(X, x_0) \cong \mathrm{P}\Pi_1^{2\bullet}(X, x_0), \tag{2}$$

where the second isomorphism is true in either version, and the first isomorphism is true for the open version and for the closed version with X a finite metric space. In either version, $\mathrm{P}\Pi_1^{\mathrm{K},\bullet}(X, x_0)$ and $\mathrm{P}\Pi_1^{\mathrm{VR},2\bullet}(X, x_0)$ have homotopy-interleaving distance zero.

Proof. The first isomorphism in Equation (2) follows from Theorem 5.9 directly. The second isomorphism is derived from Theorem 4.15, by checking the following collection of group isomorphisms

$$\pi_1^\epsilon(X, x_0) \cong \pi_1\left(\left|\mathrm{VR}_\epsilon(X)\right|, x_0\right), \forall \epsilon \geqslant 0$$

forms an isomorphism between persistent groups. □

Example 5.13 (Unit circle $\mathbb{S}^1$)**.** Recall from [AA17, Theorem 7.6] that we have homotopy equivalence

$$\left|\mathrm{VR}_r(\mathbb{S}^1)\right| \cong \begin{cases} \mathbb{S}^{2l+1}, & \text{if } \frac{l}{2l+1} 2\pi < r < \frac{l+1}{2l+3} 2\pi \text{ for some } l = 0, 1, \cdots, \\ \bigvee^{\mathfrak{c}} \mathbb{S}^{2l}, & \text{if } r = \frac{l}{2l+1} 2\pi \text{ for some } l = 0, 1, \cdots, \\ *, & \text{if } r \geqslant \pi. \end{cases}$$

Here $\mathfrak{c}$ is the cardinality of the continuum (i.e. the cardinality of $\mathbb{R}$), and $*$ is the one-point space.

Let $k \in \mathbb{Z}_{\geqslant 1}$. As $\bigvee^{\mathfrak{c}} \mathbb{S}^{2k}$ is $(2k-1)$-connected, it follows from the Hurewicz theorem (cf. [Hat01, Theorem 4.32]) that

$$\pi_{2k}\left(\bigvee^{\mathfrak{c}} \mathbb{S}^{2k}\right) \cong \mathrm{H}_{2k}\left(\bigvee^{\mathfrak{c}} \mathbb{S}^{2k}\right) \cong \mathbb{Z}^{\times \mathfrak{c}}.$$

Then the persistent homology groups of $\mathbb{S}^1$ are

$$\mathrm{PH}_n^{\mathrm{VR}}(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z}\left(\frac{k-1}{2k-1}\,2\pi,\ \frac{k}{2k+1}\,2\pi\right), & \text{if } n = 2k-1, \\ \mathbb{Z}^{\times \mathfrak{c}}\left[\frac{k}{2k+1}\,2\pi,\ \frac{k}{2k+1}\,2\pi\right], & \text{if } n = 2k. \end{cases}$$

Because $\pi_n(\mathbb{S}^m)$ is not totally known for the case $n > m$, the calculation of $\mathrm{P\Pi}_n^{\mathrm{VR}}(\mathbb{S}^1)$ can only be done for some choices of n. For example, we have

$$\mathrm{P\Pi}_n^{\mathrm{VR}}(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z}\left(0,\ \frac{2\pi}{3}\right), & \text{if } n = 1, \\ \mathbb{Z}^{\times \mathfrak{c}}\left[\frac{2\pi}{3},\ \frac{2\pi}{3}\right], & \text{if } n = 2. \end{cases}$$

5.1.2. *Persistent Rational Homotopy Groups.* The *rational homotopy groups* $\pi_n(X, x_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a pointed topological space (X, x_0) are the standard homotopy group $\pi_n(X, x_0)$ tensored with the rational numbers $\mathbb{Q}$, for each $n \geqslant 0$. We assume that all topological spaces are path-connected and denote the rational homotopy groups by $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for simplicity of notation. Compared with the difficulty of calculating homotopy groups of spheres, the computation of rational homotopy groups of spheres is substantially easier and was done by Serre in 1951 (see [Ser51]):

$$\pi_n(\mathbb{S}^{2k-1}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & n = 2k-1, \\ 0, & \text{otherwise,} \end{cases} \text{ and } \pi_n(\mathbb{S}^{2k}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & n = 2k \text{ or } n = 4k-1, \\ 0. & \text{otherwise.} \end{cases}$$

This inspires us to consider the notion of *persistent rational homotopy groups*, by tensoring persistent homotopy groups with rational numbers $\mathbb{Q}$ in the following sense.

Let the field $\Bbbk = \mathbb{Q}$. Then the category **Vec** represents the category of vector spaces over $\mathbb{Q}$. Let **Ab** be the category of Abelian groups, and let $\mathbf{PAb}^{(\mathbb{R}_+, \leqslant)}$ denote the category of functors $\mathbb{G} : (\mathbb{R}_+, \leqslant) \to \mathbf{Ab}$. Because **Ab** and **Vec** are categories of $\mathbb{Z}$-modules and $\mathbb{Q}$-modules respectively, the following defines a functor from **Ab** to **Vec**:

$$- \otimes_{\mathbb{Z}} \mathbb{Q} : G \to G \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } f \otimes_{\mathbb{Z}} \mathbb{Q} = f \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Q}} : G \otimes_{\mathbb{Z}} \mathbb{Q} \to H \otimes_{\mathbb{Z}} \mathbb{Q},$$

for each group homomorphism $f : G \to H$ of Abelian groups. Moreover, the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$ induces a functor from $\mathbf{PAb}^{(\mathbb{R}_+, \leqslant)}$ to $\mathbf{PVec}^{(\mathbb{R}_+, \leqslant)}$ such that for each $\mathbb{G} \in \mathbf{PAb}^{(\mathbb{R}_+, \leqslant)}$, $\mathbb{G} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the composition of the following two functors:

$$(\mathbb{R}_+, \leqslant) \xrightarrow{\mathbb{G}} \mathbf{Ab} \xrightarrow{- \otimes_{\mathbb{Z}} \mathbb{Q}} \mathbf{Vec}.$$

With the above construction and Example 5.13, we compute the persistent rational homotopy groups of the unit circle $\mathbb{S}^1$:

$$\mathrm{P\Pi}_n^{\mathrm{VR}}(\mathbb{S}^1) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q}\left(\frac{2k-1}{4k-1}\,2\pi,\ \frac{2k}{4k+1}\,2\pi\right) \oplus \mathbb{Q}^{\times \mathfrak{c}}\left[\frac{2k}{4k+1}\,2\pi,\ \frac{2k}{4k+1}\,2\pi\right], & \text{if } n = 4k-1, \\ \mathbb{Q}\left(\frac{2k}{4k+1}\,2\pi,\ \frac{2k+1}{4k+3}\,2\pi\right), & \text{if } n = 4k+1, \\ \mathbb{Q}^{\times \mathfrak{c}}\left[\frac{k}{2k+1}\,2\pi,\ \frac{k}{2k+1}\,2\pi\right], & \text{if } n = 2k, \end{cases}$$

as well as the persistent rational homology groups of $\mathbb{S}^1$:

$$\mathrm{PH}_n^{\mathrm{VR}}(\mathbb{S}^1;\mathbb{Q}) \cong \mathrm{PH}_n^{\mathrm{VR}}(\mathbb{S}^1)\otimes_{\mathbb{Z}}\mathbb{Q} \cong \begin{cases} \mathbb{Q}\left(\frac{k-1}{2k-1}2\pi,\ \frac{k}{2k+1}2\pi\right), & \text{if } n = 2k-1, \\ \mathbb{Q}^{\times\mathfrak{c}}\left[\frac{k}{2k+1}2\pi,\ \frac{k}{2k+1}2\pi\right], & \text{if } n = 2k. \end{cases} \tag{3}$$

The leftmost isomorphism in Equation (3) follows from the universal coefficient theorem for homology (cf. [Hat01, Theorem 3A.3]). In addition, it can be directly checked that

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_n^{\mathrm{VR}}\left(\mathbb{S}^1\right)\otimes_{\mathbb{Z}}\mathbb{Q}, \mathrm{PH}_n^{\mathrm{VR}}\left(\mathbb{S}^1;\mathbb{Q}\right)\right) = 0.$$

Note that in the case of $n = 4k-1$, because persistent modules $\mathrm{P\Pi}_n^{\mathrm{VR}}(\mathbb{S}^1)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathrm{PH}_n^{\mathrm{VR}}(\mathbb{S}^1;\mathbb{Q})$ contain different types of indecomposables, they are not isomorphic to each other.

Notice that any δ-interleaving between two persistent Abelian groups $\mathbb{G}$ and $\mathbb{H}$ induces a δ-interleaving between the persistent modules $\mathbb{G}\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathbb{H}\otimes_{\mathbb{Z}}\mathbb{Q}$. Thus,

$$d_{\mathrm{I}}\left(\mathbb{G}\otimes_{\mathbb{Z}}\mathbb{Q}, \mathbb{H}\otimes_{\mathbb{Z}}\mathbb{Q}\right) \leqslant d_{\mathrm{I}}\left(\mathbb{G},\mathbb{H}\right).$$

In addition, we have the following corollary of Theorem 6.1.

Corollary 5.14. *Let X and Y be compact chain-connected metric spaces. Then for each $n \in \mathbb{Z}_{\geqslant 2}$,*

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_n^{\mathrm{K}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}, \mathrm{P\Pi}_n^{\mathrm{K}}(Y)\otimes_{\mathbb{Z}}\mathbb{Q}\right) \leqslant d_{\mathrm{GH}}(X,Y).$$

*When $\mathrm{P\Pi}_1^{\mathrm{K}}(X), \mathrm{P\Pi}_1^{\mathrm{K}}(Y) \in$ **PAb**, we also have*

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_1^{\mathrm{K}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}, \mathrm{P\Pi}_1^{\mathrm{K}}(Y)\otimes_{\mathbb{Z}}\mathbb{Q}\right) \leqslant d_{\mathrm{GH}}(X,Y).$$

5.2. **Dendrograms and a Metric on $\pi_1(X,x_0)$.** Let $\mathbb{G} = (\{G_t\}_{t>0},\ \{f_{ts}\}_{t\leqslant s\in\mathbb{R}_+})$ be a persistent group, which is naturally an $(\mathbb{R}_+,\leqslant)$-shaped diagram in **Grp**. We can consider its limit, i.e., the group

$$G_0 := \lim\mathbb{G} = \left\{(a_t)_{t>0} \in \prod_{t>0} G_t : a_s = f_{ts}(a_t), \forall\, t\leqslant s\right\},$$

endowed with natural projections $p_t : G_0 \to G_t$ picking out the t-th component. Since $p_s = f_{ts}\circ p_t$ for $t \leqslant s$, we have $\ker(p_t)\subset\ker(p_s)$. If p_t is surjective, then $G_t \cong G_0/\ker(p_t)$ can be regarded as a partition of the limit G_0. If p_t and p_s are both surjective for some $t\leqslant s$, then, as partitions of G_0, G_t refines G_s.

By Theorem 4.19, if a metric space (X,d_X) is both u.l.p.c. and u.s.l.s.c., then each Φ_ϵ^X is surjective and $\pi_1(X)\cong\lim\pi_1^\epsilon(X)$, in which case each π_1^ϵ can be regarded as a partition of $\pi_1(X)$. Furthermore, applying Theorem 4.25 we obtain the following theorem:

Theorem 5.15 (Dendrogram over $\pi_1(X)$)**.** *Let a compact geodesic metric space X be semi-locally simply connected (s.l.s.c.). Then there is a dendrogram $\theta_{\pi_1(X)}$ over $\pi_1(X)$, given by*

$$\theta_{\pi_1(X)}(\epsilon) := \begin{cases}\pi_1^\epsilon(X), & \textit{if } \epsilon > 0\\ \pi_1(X), & \textit{if } \epsilon = 0.\end{cases}$$

As an example, let us look at the dendrogram associated to $\mathbb{S}^1$ and understand the induced metric $\mu_{\theta_{\pi_1(\mathbb{S}^1)}}$.

Example 5.16. When $r = 1$ in Example 4.13, associated to $\mathrm{P}\Pi_1(\mathbb{S}^1)$ we have a dendrogram over $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ shown in Figure 2 on Page 5. The metric $\mu_{\theta_{\pi_1(\mathbb{S}^1)}}$ induces an ultrametric d on $\mathbb{Z}$ given by

$$d(n, m) = \begin{cases} \frac{2\pi}{3}, & \text{if } n \neq m, \\ 0, & \text{if } n = m. \end{cases}$$

In Example 4.18, a simply connected compact space (e.g. the unit sphere $\mathbb{S}^2$) has a trivial associated dendrogram. Thus, the metric space $\left(\pi_1(\mathbb{S}^2), \mu_{\theta_{\pi_1(\mathbb{S}^2)}}\right)$ is (isometric to) the one-point metric space $*$. It follows from Proposition 3.5 that

$$d_{\mathrm{GH}}\left(\left(\pi_1(\mathbb{S}^1), \mu_{\theta_{\pi_1(\mathbb{S}^1)}}\right), \left(\pi_1(\mathbb{S}^2), \mu_{\theta_{\pi_1(\mathbb{S}^2)}}\right)\right) = \tfrac{1}{2}\operatorname{diam}(\pi_1(\mathbb{S}^1)) = \tfrac{\pi}{3}.$$

By the same argument, we conclude:

Corollary 5.17. *Let a compact geodesic metric space X be semi-locally simply connected (s.l.s.c.) and let Y be a simply connected compact space, then*

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) = \tfrac{1}{2}\operatorname{diam}(\pi_1(X)).$$

The next theorem, following immediately from Theorem 5.15, Proposition 3.15 and Theorem 4.19, shows that the Gromov-Hausdorff distance between fundamental groups can be estimated by the interleaving distance between dendrograms.

Theorem 5.18. *If compact geodesic metric spaces X and Y are u.l.p.c. and u.s.l.s.c., then*

$$\tfrac{1}{2}\cdot d_{\mathrm{I}}(\mathcal{V}_{\Bbbk}\circ\theta_{\pi_1(X)}, \mathcal{V}_{\Bbbk}\circ\theta_{\pi_1(Y)}) \leqslant d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leqslant d_{\mathrm{I}}(\theta_{\pi_1(X)}, \theta_{\pi_1(Y)}),$$

where

$$d_{\mathrm{I}}(\theta_{\pi_1(X)}, \theta_{\pi_1(Y)}) = d_{\mathrm{I}}^{\mathbf{Set}}(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y)) \leqslant d_{\mathrm{I}}^{\mathbf{Grp}}(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y)).$$

We finish this section with a stability result for $\mu_{\theta_{\pi_1(\bullet)}}$.

Theorem 5.19 (ℓ^∞-stability for dendrograms over $\pi_1(\bullet)$)**.** *Let compact geodesic metric spaces (X, d_X) and (Y, d_Y) be u.l.p.c. and u.s.l.s.c. Suppose X and Y are homotopy equivalent, i.e., there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X. Then*

$$\left\|\mu_{\theta_{\pi_1(X)}} - \mu_{\theta_{\pi_1(Y)}} \circ (\pi_1 f, \pi_1 f)\right\|_\infty \leqslant \max\{\operatorname{dis}(f), \operatorname{dis}(g)\},$$

where $\pi_1 f : \pi_1(X) \to \pi_1(Y)$ is the induced homomorphism. In particular, when $X = Y$ and d_X and d_Y induce the same topology on X, we can take f and g to be the identity map. In this case,

$$\left\|\mu_{\theta_{\pi_1(X)}} - \mu_{\theta_{\pi_1(Y)}}\right\|_\infty \leqslant \|d_X - d_Y\|_\infty.$$

Proof. Note that the induced homomorphisms $\pi_1 f : \pi_1(X) \to \pi_1(Y)$ and $\pi_1 g : \pi_1(Y) \to \pi_1(X)$ are isomorphisms, and these isomorphisms are inverses of each other. Since $\operatorname{dis}(\pi_1 f) = \operatorname{codis}(\pi_1 f, \pi_1 g) = \operatorname{dis}(\pi_1 g)$, we have

$$\max\{\operatorname{dis}(\pi_1 f), \operatorname{codis}(\pi_1 f, \pi_1 g), \operatorname{dis}(\pi_1 g)\} = \operatorname{dis}(\pi_1 f).$$

Let γ and γ' be two continuous loops in X based at x_0. Let $\delta > \mu_{\theta_{\pi_1(X)}}(\gamma, \gamma')$. Then there exists $\epsilon \geqslant \delta$ and ϵ-loops α and α', which are strong ϵ-loops along γ and γ' respectively, such that $\alpha \sim_1^\epsilon \alpha'$ via an ϵ-homotopy H. By an argument similar to that of Proposition 5.3, $f(\alpha)$ and $f(\alpha')$ are $(\epsilon + \mathrm{dis}(f))$-loops, and $f(H)$ is an $(\epsilon + \mathrm{dis}(f))$-homotopy. Directly checking from the definition, we see that $f(\alpha)$ and $f(\alpha')$ are strong $(\epsilon + \mathrm{dis}(f))$-loops along $\pi_1 f(\gamma)$ and $\pi_1 f(\gamma')$, respectively. Thus, $\mu_{\theta_{\pi_1(Y)}}(\pi_1 f(\gamma), \pi_1 f(\gamma')) \leqslant \epsilon + \mathrm{dis}(f)$. Letting $\delta \searrow \mu_{\theta_{\pi_1(X)}}(\gamma, \gamma')$, we obtain $\mu_{\theta_{\pi_1(Y)}}(\pi_1 f(\gamma), \pi_1 f(\gamma')) \leqslant \mu_{\theta_{\pi_1(X)}}(\gamma, \gamma') + \mathrm{dis}(f)$. Using the fact that $\pi_1 f$ and $\pi_1 g$ are inverses of each other, we can prove in a similar way that $\mu_{\theta_{\pi_1(X)}}(\gamma, \gamma') \leqslant \mu_{\theta_{\pi_1(Y)}}(\pi_1 f(\gamma), \pi_1 f(\gamma')) + \mathrm{dis}(g)$. It follows that

$$\left\| \mu_{\theta_{\pi_1(X)}} - \mu_{\theta_{\pi_1(Y)}} \circ (\pi_1 f, \pi_1 f) \right\|_\infty = \mathrm{dis}(\pi_1 f) \leqslant \max\{\mathrm{dis}(f), \mathrm{dis}(g)\}.$$

□

Remark 5.20. Despite Theorem 5.19, it is not clear whether

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right) \right) \leqslant d_{\mathrm{GH}}(X, Y). \tag{4}$$

The difficulty to prove such a claim is that arbitrary set maps do not necessarily induce homomorphisms between fundamental groups. In §6, we will prove a weaker version of Equation (4): with a factor 2 on the right hand side (cf. Theorem 6.2).

5.2.1. *An application on Riemannian manifolds.* As an application of Theorem 5.19, let us consider a connected Riemannian manifold (M, g), where M is a smooth manifold and g smoothly assigns an inner product g_p to each tangent space T_pM for each $p \in M$. Let $\lambda : M \to \mathbb{R}_+$ be a smooth positive function. Then $(M, \lambda \cdot g)$ is also a Riemmannian manifold. A change of Riemannian metric $g \to \tilde{g}$ is called a *conformal change*, if angles between two vectors with respect to g and $\tilde{g}$ are the same at every point of the manifold. It is clear that $g \to \lambda \cdot g$ is a conformal change. Let λ and $\tilde{\lambda}$ be two smooth positive functions from M to $\mathbb{R}_+$. Given any two distinct points x and x' in M, if γ is a smooth curve on M from x to x', then

$$\left| L_{\lambda g}(\gamma) - L_{\tilde{\lambda} g}(\gamma) \right| = \left| \int_\gamma \lambda g - \tilde{\lambda} g \right| \leqslant \|\lambda - \tilde{\lambda}\|_\infty \cdot \int_\gamma g,$$

where $L_g(\gamma)$ represents the length of γ in (M, g). It follows that

$$\begin{aligned} d_{\lambda g}(x, x') &\leqslant \inf_\gamma L_{\lambda g}(\gamma) + \inf_\gamma \left| L_{\lambda g}(\gamma) - L_{\tilde{\lambda} g}(\gamma) \right| \\ &\leqslant \inf_\gamma L_{\tilde{\lambda} g}(\gamma) + \|\lambda - \tilde{\lambda}\|_\infty \cdot \inf_\gamma \int_\gamma g \\ &\leqslant d_{\tilde{\lambda} g}(x, x') + \|\lambda - \tilde{\lambda}\|_\infty \cdot \mathrm{diam}(M, g). \end{aligned}$$

Thus, we have

$$\|d_{\lambda g} - d_{\tilde{\lambda} g}\|_\infty \leqslant \|\lambda - \tilde{\lambda}\|_\infty \cdot \mathrm{diam}(M, g). \tag{5}$$

Since $(M, \lambda \cdot g)$ and $(M, \tilde{\lambda} \cdot g)$ have the same homotopy type, we can apply Theorem 5.19 and Equation 5 to obtain

Corollary 5.21.

$$\left\| \mu_{\theta_{\pi_1(M, \lambda g)}} - \mu_{\theta_{\pi_1(M, \tilde{\lambda} g)}} \right\|_\infty \leqslant \|\lambda - \tilde{\lambda}\|_\infty \cdot \mathrm{diam}(M, g).$$

For example, let us take M to be the 2-dimensional torus T^2 and g to be the flat metric on it. In other words, (T^2, g) is the quotient space of the Euclidean square $[0, 2\pi] \times [0, 2\pi]$ by identifying the opposite sides (see Figure 3). Then we have

$$\left\| \mu_{\theta_{\pi_1(T^2,\lambda g)}} - \mu_{\theta_{\pi_1(T^2,\tilde{\lambda} g)}} \right\|_\infty \leqslant \|\lambda - \tilde{\lambda}\|_\infty \cdot \sqrt{5}\pi,$$

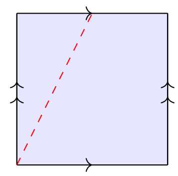


FIGURE 3. The manifold (T^2, g), as the quotient space of the Euclidean square $[0, 2\pi] \times [0, 2\pi]$ by identifying the opposite sides. The length of the red line realizes the diameter of T^2.

5.3. **Persistent Homotopy Groups under Products.** Let the Cartesian product $X \times Y$ of two metric spaces be equipped with the ℓ^∞ product metric:

$$d_{X\times Y}((x, y), (x', y')) := \max\{d_X(x, x'), d_Y(y, y')\}, \forall x, x' \in X; y, y' \in Y.$$

Note that when X and Y are chain-connected, $X \times Y$ is also chain-connected.

Proposition 5.22 (Proposition 10.2, [AA17])**.** *Let X and Y be pointed metric spaces. For each $\epsilon > 0$, we have the basepoint preserving homotopy equivalence*

$$|\mathrm{VR}_\epsilon(X)| \times |\mathrm{VR}_\epsilon(Y)| \xrightarrow{\cong} |\mathrm{VR}_\epsilon(X \times Y)| .$$

Furthermore, for $0 < \epsilon \leqslant \epsilon'$, we have the following commutative diagram:

$$\begin{array}{ccc} |\mathrm{VR}_\epsilon(X)| \times |\mathrm{VR}_\epsilon(Y)| & \hookrightarrow & |\mathrm{VR}_{\epsilon'}(X)| \times |\mathrm{VR}_{\epsilon'}(Y)| \\ \cong \downarrow & & \downarrow \cong \\ |\mathrm{VR}_\epsilon(X \times Y)| & \hookrightarrow & |\mathrm{VR}_{\epsilon'}(X \times Y)| . \end{array}$$

Recall from [Hat01, Proposition 4.2] the fact that $\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$. Thus, by applying the homotopy group functor to the above commuting diagram where all maps preserve basepoints, we obtain the following corollary:

Corollary 5.23. *Let (X, x_0) and (Y, y_0) be pointed metric spaces. There is a natural isomorphism of persistent groups:*

$$\mathrm{P}\Pi_n^{\mathrm{VR}}(X \times Y, (x_0, y_0)) \cong \mathrm{P}\Pi_n^{\mathrm{VR}}(X, x_0) \times \mathrm{P}\Pi_n^{\mathrm{VR}}(Y, y_0),$$

the product of $\mathrm{P}\Pi_n^{\mathrm{VR}}(X, x_0)$ and $\mathrm{P}\Pi_n^{\mathrm{VR}}(Y, y_0)$ (see Page 9). It follows that

$$\mathrm{P}\Pi_1(X \times Y, (x_0, y_0)) \cong \mathrm{P}\Pi_1(X, x_0) \times \mathrm{P}\Pi_1(Y, y_0). \tag{6}$$

Remark 5.24. Recall from Theorem 5.15 that if a compact geodesic metric space X is both u.l.p.c. and u.s.l.s.c., then there is a dendrogram $\theta_{\pi_1(X)}$ over $\pi_1(X)$. Suppose Y is also such a space with a dendrogram $\theta_{\pi_1(Y)}$ over $\pi_1(Y)$. Since $X \times Y$ is also geodesic, u.l.p.c. and u.s.l.s.c., the left hand side of Equation (6) induces a dendrogram $\theta_{\pi_1(X\times Y)}$ over $\pi_1(X \times Y)$. Meanwhile, the product of dendrograms $\theta_{\pi_1(X)} \times \theta_{\pi_1(Y)}$ gives a dendrogram over

$\pi_1(X) \times \pi_1(Y) \cong \pi_1(X \times Y)$, cf. Page 17. In particular, the isomorphism given in Equation (6) implies that for each $\epsilon \geqslant 0$ one has

$$\theta_{\pi_1(X\times Y)}(\epsilon) = \left(\theta_{\pi_1(X)} \times \theta_{\pi_1(Y)}\right)(\epsilon) := \begin{cases} \pi_1^\epsilon(X) \times \pi_1^\epsilon(Y), & \text{if } \epsilon > 0 \\ \pi_1(X) \times \pi_1(Y), & \text{if } \epsilon = 0, \end{cases}$$

as partitions of $\pi_1(X \times Y)$.

Example 5.25. Consider the torus $T^2(r_1, r_2) := \mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$, where $0 < r_1 \leqslant r_2$ and $\mathbb{S}^1(r_1)$ and $\mathbb{S}^1(r_2)$ are both endowed with their corresponding geodesic metrics. Then

$$\mathrm{P}\Pi_1\left(\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)\right) = \mathbb{Z}\left(0, \tfrac{2\pi}{3} r_1\right) \times \mathbb{Z}\left(0, \tfrac{2\pi}{3} r_2\right) = (\mathbb{Z} \times \mathbb{Z})\left(0, \tfrac{2\pi}{3} r_1\right) \times \mathbb{Z}\left[\tfrac{2\pi}{3} r_1, \tfrac{2\pi}{3} r_2, \right)$$

and the dendrogram associated to $\mathrm{P}\Pi_1(T^2(r_1, r_2))$ over $\pi_1(T^2(r_1, r_2)) \cong \mathbb{Z} \times \mathbb{Z}$ is described in Figure 4.

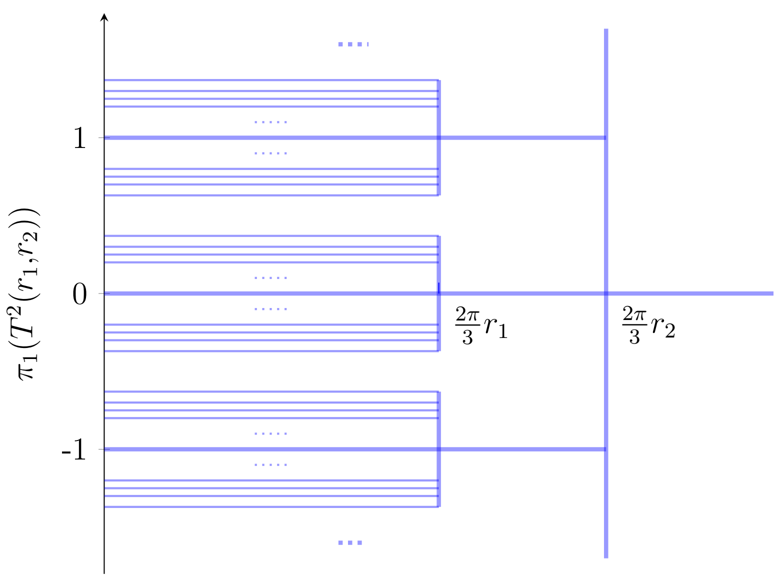


FIGURE 4. The dendrogram associated to $\mathrm{P}\Pi_1(T^2(r_1, r_2))$. The y-axis represents elements of $\pi_1(T^2(r_1, r_2)) \cong \mathbb{Z}\gamma_1 \times \mathbb{Z}\gamma_2$, where γ_1 and γ_2 are generators of $\pi_1(\mathbb{S}^1(r_1))$ and $\pi_1(\mathbb{S}^1(r_2))$, respectively. More precisely, for $k \geqslant 1$ and $l \in \mathbb{Z}$, the element $(\gamma_1^{\pm k}, \gamma_2^l)$ is represented by the y-value $l \pm \frac{1}{k+2}$ and the element (γ_1^0, γ_2^l) is represented by the y-value l. Notice that the dendrogram associated to $\mathrm{P}\Pi_1(T^2(r_1, r_2))$ contains the dendrogram associated to $\mathrm{P}\Pi_1(\mathbb{S}^1(r_2))$.

When $r_1 = r_2 = 1$, we write $T^2 := T^2(1, 1)$ and apply Example 5.13 to compute:

$$\mathrm{P}\Pi_n^{\mathrm{VR}}(T^2) \cong \begin{cases} (\mathbb{Z} \times \mathbb{Z})\left(0, \frac{2\pi}{3}\right), & \text{if } n = 1, \\ (\mathbb{Z}^{\times \mathfrak{c}} \times \mathbb{Z}^{\times \mathfrak{c}})\left[\frac{2\pi}{3}, \frac{2\pi}{3}\right], & \text{if } n = 2. \end{cases}$$

As for the persistent homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$, because $\mathrm{PH}_i^\epsilon(\mathbb{S}^1)$ is torsion free for each $i \geqslant 0$ and $\epsilon > 0$, the Künneth Theorem (cf. [Hat01, Theorem 3B.5]) can be applied to calculate $\mathrm{PH}_n^{\mathrm{VR},\epsilon}(T^2) \cong \bigoplus_{i=0}^n \mathrm{PH}_i^\epsilon(\mathbb{S}^1) \times \mathrm{PH}_{n-i}^\epsilon(\mathbb{S}^1)$. On the other hand, if the homology groups $\mathrm{PH}_n^{\mathrm{VR},\epsilon}(T^2; \mathbb{R})$ are computed with coefficients in $\mathbb{R}$, then for all integers $k \geqslant 0$, the corresponding undecorated persistence diagrams of T^2 are (cf. [LMO20, Example 4.3]):

$$\begin{aligned}
\mathrm{dgm}_{2k+1}(T^2) &= \left\{ \left(\tfrac{k}{2k+1}\,2\pi,\ \tfrac{k+1}{2k+3}\,2\pi\right), \left(\tfrac{k}{2k+1}\,2\pi,\ \tfrac{k+1}{2k+3}\,2\pi\right) \right\},\\
\mathrm{dgm}_{4k+2}(T^2) &= \left\{ \left(\tfrac{k}{2k+1}\,2\pi,\ \tfrac{k+1}{2k+3}\,2\pi\right) \right\},\\
\mathrm{dgm}_{4k+4}(T^2) &= \varnothing.
\end{aligned}$$

5.4. **Persistent Fundamental Groups under Wedge Sums.** The wedge sum $X \vee Y$ of two pointed metric spaces X and Y is the quotient space of the disjoint union of X and Y by the identification of basepoints $x_0 \sim y_0$: $X \vee Y = (X \amalg Y)/\sim$. Denote the resulting basepoint of $X \vee Y$ by z_0. Let the wedge product be equipped with the *gluing metric* (see [AAG$^+$19]):

$$d_{X\vee Y}(x,y) := d_X(x,x_0) + d_Y(y,y_0), \forall x \in X, y \in Y$$

and $d_{X\vee Y}|_{X\times X} = d_X, d_{X\vee Y}|_{Y\times Y} = d_Y$. Notice that the above definition can be generalized to the case when we glue n different pointed metric spaces $(X_1,x_1),\cdots,(X_n,x_n)$ by identifying $x_i \sim x_j$ for all $1 \leqslant i,j \leqslant n$. We denote the resulting space by $\bigvee_{i=1}^n (X_i,x_i)$, or $\bigvee_{i=1}^n X_i$ for simplicity, with the gluing metric:

$$d_{\bigvee_{i=1}^n X_i}(y_i,y_j) := d_{X_i}(y_i,x_i) + d_{X_j}(y_j,x_j), \forall i \neq j, y_i \in X_i, y_j \in X_j$$

and $d_{X_i\vee X_i}|_{X_i\times X_i} = d_{X_i}$ for $1 \leqslant i \leqslant n$.

Proposition 5.26 (Proposition 3.7, [AAG$^+$19])**.** *Let X and Y be pointed metric spaces. For each $\epsilon > 0$, we have the basepoint preserving homotopy equivalence*

$$|\mathrm{VR}_\epsilon(X)| \vee |\mathrm{VR}_\epsilon(Y)| \xrightarrow{\cong} |\mathrm{VR}_\epsilon(X \vee Y)|.$$

Furthermore, for $0 < \epsilon \leqslant \epsilon'$, we have the following commutative diagram:

$$\begin{array}{ccc}
|\mathrm{VR}_\epsilon(X)| \vee |\mathrm{VR}_\epsilon(Y)| & \hookrightarrow & |\mathrm{VR}_{\epsilon'}(X)| \vee |\mathrm{VR}_{\epsilon'}(Y)| \\
\cong\downarrow & & \downarrow\cong \\
|\mathrm{VR}_\epsilon(X\vee Y)| & \hookrightarrow & |\mathrm{VR}_{\epsilon'}(X\vee Y)|.
\end{array}$$

By applying the fundamental group functor to the above commuting diagram, where all maps preserve basepoints, we obtain a corollary about the persistent fundamental group of the wedge sum of metric spaces.

Corollary 5.27. *Let (X,x_0) and (Y,y_0) be pointed metric spaces, and let z_0 be the basepoint in $X \vee Y$ obtained by identifying x_0 and y_0. There is a natural isomorphism of persistent groups:*

$$\mathrm{P\Pi}_1(X \vee Y, z_0) \cong \mathrm{P\Pi}_1(X,x_0) * \mathrm{P\Pi}_1(Y,y_0),$$

the coproduct of $\mathrm{P\Pi}_1(X)$ and $\mathrm{P\Pi}_1(Y)$ given by componentwise free product (see Page 9).

Proof. This follows immediately from Theorem 4.15, Proposition 5.26 and the van-Kampen theorem (see [Hat01, Theorem 1.20]). □

Remark 5.28. Via the isomorphism between $\mathrm{P\Pi}_n^{\mathrm{VR}}$ and $\mathrm{P\Pi}_n^{\mathrm{K}}$, Corollary 5.23 and Corollary 5.27 can both be derived in a much simpler way using Kuratowski filtrations. See the proof of [LMO20, Theorem 4.1] for details.

Example 5.29 (Bouquets of circles)**.** Let $0 =: r_0 < r_1 < r_2 < \cdots < r_n$ and let $k_1, k_2, \cdots, k_n$ be positive integers. Consider a *bouquet of circles* given as follows: let

$$C = \left(\bigvee^{k_1} \mathbb{S}^1(r_1)\right) \vee \left(\bigvee^{k_2} \mathbb{S}^1(r_2)\right) \vee \cdots \vee \left(\bigvee^{k_n} \mathbb{S}^1(r_n)\right)$$

by gluing one point from each circle together. Then,

$$\mathrm{P\Pi}_1^\epsilon(C) = \begin{cases} \mathbb{Z}^{*(k_i+\cdots+k_n)}, & \text{if } \frac{2\pi}{3} r_{i-1} \leqslant \epsilon < \frac{2\pi}{3} r_i, \text{ for } i = 1, \cdots, n. \\ 0, & \text{if } \epsilon \geqslant \frac{2\pi}{3} r_n. \end{cases},$$

where $\mathrm{P\Pi}_1^0(C) := \pi_1(C) = \mathbb{Z}^{*(k_1+\cdots+k_n)}$. In addition,

$$\mathrm{P\Pi}_1(C) = \coprod_{i=1}^n \mathbb{Z}^{*k_i} \left[0, \tfrac{2\pi}{3} r_i\right) = \coprod_{i=1}^n \mathbb{Z}^{*(k_i+\cdots+k_n)} \left[\tfrac{2\pi}{3} r_{i-1}, \tfrac{2\pi}{3} r_i\right).$$

Notice that the number of free generators of $\mathrm{P\Pi}_1^\epsilon(C)$ can be represented by a stair-case function:

$$f_C(\epsilon) := \begin{cases} k_i + \cdots + k_n, & \text{if } \epsilon \in \left[\frac{2\pi}{3} r_{i-1}, \frac{2\pi}{3} r_i\right), \text{ for } i = 1, \cdots, n. \\ 0, & \text{if } \epsilon \in \left[\frac{2\pi}{3} r_n, +\infty\right). \end{cases}$$

Taking $r_1 = 1, r_2 = 2$ and $r_3 = 3$, we obtain the function f_C shown in Figure. 5

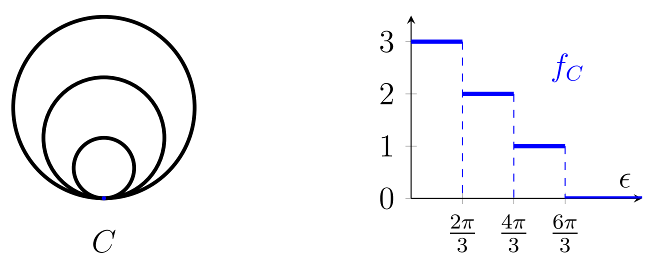


FIGURE 5. Bouquets of three circles with $r_1 = 1, r_2 = 2$ and $r_3 = 3$ (left) and the corresponding function f_C (right) representing the number of free generators of $\mathrm{P\Pi}_1^\epsilon(C)$.

5.5. **Relation to Persistent Homology Groups: Persistent Hurewicz Theorem.** In [BCW14], Barcelo et al. established a slightly different definition of ϵ-homotopy, where an ϵ-homotopy between two ϵ-chains γ_1 and γ_2 (of the same size n) is an ϵ-Lipschitz map $H : ([n] \times [m], \ell^1) \to (X, d_X)$, instead of equipping $[n] \times [m]$ with the ℓ^∞ metric as we do in this paper. They proved a discrete version of the Hurewicz theorem under the ℓ^1 metric in [BCW14]. Via a similar argument, we obtain the following:

Theorem 5.30 (Discrete Hurewicz Theorem)**.** *Let $\epsilon > 0$. Let (X, x_0) be a pointed ϵ-connected metric space. Then there is a surjective group homomorphism*

$$\rho_\epsilon : \pi_1^\epsilon(X, x_0) \twoheadrightarrow \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X) := \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X; \mathbb{Z})$$

such that $\ker(\rho_\epsilon) = [\pi_1^\epsilon(X, x_0), \pi_1^\epsilon(X, x_0)]$*, the commutator group of* $\pi_1^\epsilon(X, x_0)$*.*

By checking that $\rho := \{\rho_\epsilon\}_{\epsilon > 0} \in \mathrm{Hom}(\mathrm{P\Pi}_1(X), \mathrm{PH}_1^{\mathrm{VR}}(X))$ (see Page 39) and applying the ismorphism theorem of persistent fundamental groups, we obtain the persistent Hurewicz theorem:

Theorem 5.31 (Persistent Hurewicz Theorem)**.** *Let X be a chain-connected metric space. Then there is a natural transformation*

$$\mathrm{P\Pi}_1^{\mathrm{K}}(X) \overset{\rho}{\Rightarrow} \mathrm{PH}_1^{\mathrm{K}}(X),$$

where for each $\epsilon > 0$, ρ_ϵ is surjective and $\ker(\rho_\epsilon)$ is the commutator group of $\mathrm{P\Pi}_1^{\mathrm{K},\epsilon}(X)$.

Proof of Theorem 5.30. Here we adapt the proof of Theorem 4.1 from [BCW14]. For simplicity, we fix ϵ and write ρ for ρ_ϵ. Also, we write d for d_X and omit $*$ for the concatenation of discrete loops.

Let $[\gamma] \in \pi_1^\epsilon(X, x_0)$ and choose a representative $\gamma = x_0x_1 \cdots x_{n+1}$ with $x_{n+1} = x_0$. For each $i \in [n]$, since $d(x_i, x_{i+1}) \leqslant \epsilon$, $\sigma_i := \{x_i, x_{i+1}\}$ is an 1-simplex in $\mathrm{VR}_\epsilon(X)$, implying that $\sum_i \sigma_i \in C_1(\mathrm{VR}_\epsilon(X); \mathbb{Z})$. Note that $\sum_i \sigma_i \in \ker(\partial_1)$, because

$$\partial_1 \left(\sum_i \sigma_i \right) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n+1} - x_n) = 0.$$

Define $\tilde{\rho}(\gamma) = \sum_i \sigma_i$. Let $p : \ker(\partial_1) \to \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X)$ be the canonical projection, and define

$$\rho([\gamma]) := p(\tilde{\rho}(\gamma)).$$

Claim 1: ρ is well defined.

It suffices to prove that ρ is well-defined under basic moves. Let $\gamma = x_0x_1 \cdots x_nx_0$. Suppose that by removing some point x_i ($1 \leqslant i \leqslant n$), we obtain an ϵ-chain $\gamma_i = x_0x_1 \cdots x_{i-1}x_{i+1} \cdots x_nx_0$. Then,

$$\tilde{\rho}(\gamma) - \tilde{\rho}(\gamma_i) = \{x_{i-1}, x_i\} + \{x_i, x_{i+1}\} - \{x_{i-1}, x_{i+1}\} = \partial_2(\{x_{i-1}, x_i, x_{i+1}\}).$$

Because γ_i is an ϵ-chain, $d(x_{i-1}, d_{i+1}) \leqslant \epsilon$, and thus $\{x_{i-1}, x_i, x_{i+1}\} \in C_2(\mathrm{VR}_\epsilon(X); \mathbb{Z})$. It follows that $\tilde{\rho}(\gamma) - \tilde{\rho}(\gamma_i) \in \mathrm{Im}(\partial_2)$ and $\rho(\gamma) = \rho(\gamma_i)$.

If adding y_i results a new ϵ-chain $\gamma_i' = x_0x_1 \cdots x_{i-1}y_ix_i \cdots x_nx_0$, we can apply similar arguments to obtain $\tilde{\rho}(\gamma) - \tilde{\rho}(\gamma_i') = \partial_2(\{x_{i-1}, y_i, x_i\}) \in \mathrm{Im}(\partial_2)$ and thus $\rho(\gamma) = \rho(\gamma_i')$.

Claim 2: ρ is a group homomorphism.

Let $\gamma_1 = x_0x_1 \cdots x_nx_{n+1}$ and $\gamma_2 = y_0y_1 \cdots y_my_{m+1}$ where $x_0 = x_{n+1} = y_0 = y_{m+1}$. Then

$$\tilde{\rho}(\gamma_1\gamma_2) = \sum_{i=0}^{n} \{x_i, x_{i+1}\} + \sum_{j=0}^{m} \{y_j, y_{j+1}\} = \tilde{\rho}(\gamma_1) + \tilde{\rho}(\gamma_2) \implies \rho(\gamma_1\gamma_2) = \rho(\gamma_1) + \rho(\gamma_2).$$

Claim 3: ρ is surjective.

Let $\lambda \in \ker(\partial_1)$. As an element of $C_1(\mathrm{VR}_\epsilon(X); \mathbb{Z})$, λ can be written as $\sum_{i=1}^k n_i\sigma_i$ for some $n_i \in \mathbb{Z}$ and $\sigma_i = \{x_i, y_i\}$ such that $d(x_i, y_i) \leqslant \epsilon$ for all i. Since $\partial_1(\lambda) = 0$, we have

$$\sum_{i=1}^{k} n_i(y_i - x_i) = 0. \tag{7}$$

Let $S = \{x_i, y_i : i = 1, \cdots, k\}$. Given $q \in S$,

- let m_q be the sum of coefficients of q in Equation (7). Note that $m_q = 0$.
- let β_q be an ϵ-chain from x_0 to q in X.

For each i, define $\eta_i := \beta_{x_i}\sigma_i\beta_{y_i}^{-1}$. Clearly, every η_i is an ϵ-loop. Let $\gamma = \eta_1^{n_1}\eta_2^{n_2}\cdots\eta_k^{n_k}$, which is again an ϵ-loop. In addition, we have $\rho([\gamma]) = \lambda$, because

$$\begin{aligned}\tilde{\rho}(\gamma) &= \sum_{i=1}^{k} n_i\left(\tilde{\rho}(\beta_{x_i}) + \sigma_i - \tilde{\rho}(\beta_{y_i})\right)\\ &= \sum_{i=1}^{k} n_i\sigma_i + \sum_{i=1}^{k} n_i\left(\tilde{\rho}(\beta_{x_i}) - \tilde{\rho}(\beta_{y_i})\right)\\ &= \sum_{i=1}^{k} n_i\sigma_i + \sum_{q\in S} m_q\tilde{\rho}(\beta_q)\\ &= \sum_{i=1}^{k} n_i\sigma_i = \lambda.\end{aligned}$$

Claim 4: $\ker(\rho) = [\pi_1^\epsilon(X,x_0), \pi_1^\epsilon(X,x_0)]$.

By the previous claims, $\mathrm{Im}(\rho) = \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X)$ is Abelian. Since $\mathrm{Im}(\rho) \cong \pi_1^\epsilon(X,x_0)/\ker(\rho)$, it must be true that $[\pi_1^\epsilon(X,x_0), \pi_1^\epsilon(X,x_0)] \subset \ker(\rho)$. It remains to prove the reverse inclusion. Let $\gamma = v_0v_1\cdots v_rv_{r+1}$, with $v_0 = v_{r+1} = x_0$, be an ϵ-loop such that $\gamma \in \ker(\rho)$. It follows from $\rho(\gamma) = 0$ that there exists $\sigma = \sum_{i=1}^k n_i\sigma_i$ with $\sigma_i = \{x_i, y_i, z_i\} \in C_2(\mathrm{VR}_\epsilon(X);\mathbb{Z})$ such that

$$\gamma = \partial_2(\sigma) = \sum_{i=1}^{k} n_i\left(\{x_i,y_i\} + \{z_i,x_i\} + \{y_i,z_i\}\right). \tag{8}$$

Denote $\sigma_i^1 = \{x_i,y_i\}, \sigma_i^2 = \{y_i,z_i\}$ and $\sigma_i^3 = \{z_i,x_i\}$.

- Let $L := \{\sigma_i^j : i = 1,\cdots,k; j = 1,2,3\}$. Given $\zeta \in L$, let m_ζ be the sum of coefficients of ζ in Equation (8).
- Let S be the set of endpoints of all σ_i^j. For each $q \in S$, let β_q be an ϵ-chain from x_0 to q in X.

For each i, define $\eta_i := (\beta_{x_i}\sigma_i^1\beta_{y_i}^{-1})(\beta_{y_i}\sigma_i^2\beta_{z_i}^{-1})(\beta_{z_i}\sigma_i^3\beta_{x_i}^{-1}) = \beta_{x_i}\sigma_i^1\sigma_i^2\sigma_i^3\beta_{x_i}^{-1}$, which is an ϵ-chain. Assume $\beta_{x_i} = x_0w_1\cdots w_lx_i$. It follows from $\{x_i,y_i,z_i\} \in C_2(\mathrm{VR}_\epsilon(X);\mathbb{Z})$ that

$$\begin{aligned}\eta_i &= x_0w_1\cdots w_lx_ix_iy_iy_iz_iz_ix_iw_l\cdots w_1x_0\\ &\sim_1^\epsilon x_0w_1\cdots w_lx_iy_iz_ix_iw_l\cdots w_1x_0\\ &\sim_1^\epsilon x_0w_1\cdots w_lx_iy_ix_iw_l\cdots w_1x_0\\ &\sim_1^\epsilon x_0w_1\cdots w_lx_ix_iw_l\cdots w_1x_0\\ &\sim_1^\epsilon x_0.\end{aligned}$$

Let $\eta = \eta_1^{n_1}\eta_2^{n_2}\cdots\eta_k^{n_k}$, which is ϵ-homotopic to the constant loop x_0.

Recall that $\gamma = v_0v_1\cdots v_rv_{r+1}$. Let $\tau_i = \{v_i, v_{i+1}\}$ for $i \in [r]$ and let $\tau = \prod_{i=0}^r \beta_{\tau_i(0)}\tau_i\beta_{\tau_i(1)}^{-1}$, which is ϵ-homotopic to γ. Therefore, $\gamma \sim_1^\epsilon \tau\eta^{-1}$. For each $\zeta \in L$, the loop $\beta_{\zeta(0)}\zeta\beta_{\zeta(1)}^{-1}$ appears m_ζ times in τ, and $-m_\zeta$ times in η^{-1}. So each $\beta_{\zeta(0)}\zeta\beta_{\zeta(1)}^{-1}$ appears in $\tau\eta^{-1}$ with exponent adding up to zero, so $[\gamma]_\epsilon = [\tau\eta^{-1}]_\epsilon \in [\pi_1^\epsilon(X,x_0), \pi_1^\epsilon(X,x_0)]$. □

Proof of Theorem 5.31. We follow the notations from the proof of Theorem 5.30, but adapt them to indicate their dependence on the index ϵ. For $\epsilon > 0$, we let ∂_1^ϵ be the first boundary map of chain complex of $\mathrm{VR}_\epsilon(X)$, let p^ϵ be the canonical projection $\ker(\partial_1^\epsilon) \to \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X)$ and let $\tilde{\rho}^\epsilon : \mathcal{L}^\epsilon(X, x_0) \to \ker(\partial_1^\epsilon)$ be given by

$$x_0 x_1 \cdots x_n x_0 \mapsto (x_0, x_1) + (x_1, x_2) + \cdots + (x_n, x_0).$$

To prove the theorem, it suffices to check that $\{\rho_\epsilon\}_{\epsilon>0} \in \mathrm{Hom}(\mathrm{P\Pi}_1(X), \mathrm{PH}_1^{\mathrm{VR}}(X))$. For $0 < \epsilon \leqslant \epsilon'$, it can be directly checked that each of the three squares in the following diagram commutes:

$$\begin{array}{ccccccc}
\mathrm{P\Pi}_1^{\epsilon'}(X) & \hookrightarrow & \mathcal{L}^{\epsilon'}(X) & \xrightarrow{\tilde{\rho}^{\epsilon'}} & \ker(\partial_1^{\epsilon'}) & \xrightarrow{p^{\epsilon'}} & \mathrm{PH}_1^{\mathrm{VR},\epsilon'}(X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{P\Pi}_1^{\epsilon}(X) & \hookrightarrow & \mathcal{L}^{\epsilon}(X) & \xrightarrow[\tilde{\rho}^{\epsilon}]{} & \ker(\partial_1^{\epsilon}) & \xrightarrow[p^{\epsilon}]{} & \mathrm{PH}_1^{\mathrm{VR},\epsilon}(X).
\end{array}$$

□

Remark 5.32. By the isomorphism of persistent fundamental groups (cf. Theorem 5.12), Theorem 5.30 follows from the standard Hurewicz theorem. Futhermore, for any $n \geqslant 1$, there exists a group homomorphism

$$\mathrm{P\Pi}_n^{\mathrm{K},\epsilon}(X, x_0) \xrightarrow{h_n} \mathrm{PH}_n^{\mathrm{K},\epsilon}(X, x_0).$$

When X^ϵ is such that $\pi_i(X^\epsilon, x_0) = 0$ for $1 \leqslant i \leqslant n-1$, the homomorphism h_n is an isomorphism.

6. Stability of Persistent Homotopy Groups

Recall that the interleaving distance between persistent groups $\mathbb{G}$ and $\mathbb{H}$ is given in Definition 3.8 by letting $\mathcal{C} = \mathbf{Grp}$:

$$d_{\mathrm{I}}(\mathbb{G}, \mathbb{H}) := \inf\{\delta \geqslant 0 : \mathbb{G} \text{ and } \mathbb{H} \text{ are } \delta\text{-interleaved}\}.$$

In this section, we prove the following stability theorem for the interleaving distance of persistent K-homotopy groups (equivalently, for persistent VR-homotopy groups):

Theorem 6.1 (d_{I}-stability for $\mathrm{P\Pi}_n^{\mathrm{K}}(\bullet)$ and $\mathrm{P\Pi}_n^{\mathrm{VR}}(\bullet)$)**.** *Let (X, x_0) and (Y, y_0) be pointed compact metric spaces. Then, for each $n \in \mathbb{Z}_{\geqslant 1}$,*

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_n^{\mathrm{K}}(X, x_0), \mathrm{P\Pi}_n^{\mathrm{K}}(Y, y_0)\right) \leqslant d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

If X and Y are chain-connected, then

$$d_{\mathrm{I}}\left(\mathrm{P\Pi}_n^{\mathrm{K}}(X), \mathrm{P\Pi}_n^{\mathrm{K}}(Y)\right) \leqslant d_{\mathrm{GH}}(X, Y).$$

Via the isomorphism $\mathrm{P\Pi}_n^{\mathrm{K},\bullet} \cong \mathrm{P\Pi}_n^{\mathrm{VR},2\bullet}$, the above two inequalities also hold for $\mathrm{P\Pi}_n^{\mathrm{VR}}(\bullet)$, up to a factor $\frac{1}{2}$.

As a consequence of the above theorem, we can derive a stability theorem for fundamental groups:

Theorem 6.2 (d_{GH}-stability for $\theta_{\pi_1(\bullet)}$)**.** *If compact geodesic metric spaces X and Y are s.l.s.c., then*

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leqslant 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Given the isomorphism between the persistent fundamental group and the persistent K-fundamental group (cf. Theorem 5.12), another immediate application of Theorem 6.1 is the stability for persistent fundamental groups:

Theorem 6.3 (d_{I}-stability for $\mathrm{P}\Pi_1(\bullet)$)**.** *Let (X, x_0) and (Y, y_0) be pointed compact metric spaces. Then*

$$d_{\mathrm{I}}(\mathrm{P}\Pi_1(X, x_0), \mathrm{P}\Pi_1(Y, y_0)) \leqslant 2 \cdot d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

6.1. **Stability for $\mathrm{P}\Pi_n^{\mathrm{K}}(\bullet)$, $\mathrm{P}\Pi_n^{\mathrm{VR}}(\bullet)$ and $\pi_1(\bullet)$.** To prove Theorem 6.1, we use the stability of homotopy interleaving distance under the Gromov-Hausdorff distance (see Theorem 3.12). As homotopy groups depend on basepoints when spaces are not connected, all maps between topological spaces are required to be basepoint preserving in this section.

Given an $\mathbb{R}_+$-space $\mathbb{V} : (\mathbb{R}_+, \leqslant) \to \mathbf{Top}$, we denote by $\lim \mathbb{V}$ the limit of $\mathbb{V}$ together with morphisms $v_t : \lim \mathbb{V} \to \mathbb{V}(t)$ for each $t \in \mathbb{R}_+$. For any $x \in \lim \mathbb{V}$, let $(\mathbb{V}, x)$ be the functor from $(\mathbb{R}_+, \leqslant)$ to $\mathbf{Top}^*$ such that $(\mathbb{V}, x)(t) = (V_t, v_t(x))$ and $(\mathbb{V}, x)(t \leqslant s) = v_t^s$. Given two $\mathbb{R}_+$-spaces $\mathbb{V}$ and $\mathbb{W}$, as well as $x \in \lim \mathbb{V}$ and $y \in \lim \mathbb{W}$, we say that $(\mathbb{V}, x)$ and $(\mathbb{W}, y)$ are weakly equivalent if $\mathbb{V} \simeq \mathbb{W}$ via basepoint preserving maps. And $(\mathbb{V}, x)$ and $(\mathbb{W}, y)$ are said to be δ-homotopy-interleaved if there exist $\mathbb{V}', \mathbb{W}' : (\mathbb{R}_+, \leqslant) \to \mathbf{Top}^*$ such that $(\mathbb{V}', x') \simeq (\mathbb{V}, x)$, $(\mathbb{W}', y') \simeq (\mathbb{W}, y)$ for some $x' \in \lim \mathbb{V}'$ and $y' \in \lim \mathbb{W}'$, and $(\mathbb{V}', x')$ and $(\mathbb{W}', y')$ are δ-interleaved.

Lemma 6.4. *Let $\mathbb{V}$ and $\mathbb{W}$ be two $\mathbb{R}_+$-spaces. If $\mathbb{V} \simeq \mathbb{W}$, then there exists $x \in \lim \mathbb{V}$ and $y \in \lim \mathbb{W}$ such that $(\mathbb{V}, x) \simeq (\mathbb{W}, y)$. Moreover, if $(\mathbb{V}, x) \simeq (\mathbb{W}, y)$, then for all $n \geqslant 1$,*

$$\pi_n \circ (\mathbb{V}, x) \cong \pi_n \circ (\mathbb{W}, y).$$

Proof. Since $\mathbb{V} \simeq \mathbb{W}$, there exist an $\mathbb{R}_+$-space $\mathbb{U}$ and natural transformations $f : \mathbb{U} \Rightarrow \mathbb{V}$ and $g : \mathbb{U} \Rightarrow \mathbb{W}$ that are (objectwise) weak equivalences. Because of the universal property of $\lim \mathbb{V}$, there exists a morphism $\lim f$ such that the following diagram commutes for all $0 < t \leqslant s$:

$$\begin{array}{ccccc} V_t & & \xleftarrow[\simeq]{f_t} & & U_t \\ \downarrow & \nwarrow & & \nearrow & \downarrow \\ & \lim \mathbb{V} & \xleftarrow{\lim f} & \lim \mathbb{U} & \\ \downarrow & \swarrow & & \searrow & \downarrow \\ V_s & & \xleftarrow[\simeq]{f_s} & & U_s. \end{array}$$

Similarly, there exists a morphism $\lim g : \lim \mathbb{U} \to \lim \mathbb{W}$ satisfying a similar commutative diagram. Take any $z \in \lim \mathbb{U}$ and let $x := (\lim f)(z)$ and $y := (\lim g)(z)$. It is clear that $(\mathbb{V}, x) \simeq (\mathbb{W}, y)$.

In addition, we have the following commuting diagrams (for each $0 < t \leqslant s$):

$$\begin{array}{ccccc} \pi_n(V_t, v_t(x)) & \xleftarrow[\cong]{\pi_n(f_t)\circ(\pi_n(g_t))^{-1}} & \pi_n(W_t, w_t(y)) & \xrightarrow[\cong]{\pi_n(g_t)\circ(\pi_n(f_t))^{-1}} & \pi_n(V_t, v_t(x)) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_n(V_s, v_s(x)) & \xleftarrow[\pi_n(f_s)\circ(\pi_n(g_s))^{-1}]{\cong} & \pi_n(W_s, w_s(y)) & \xrightarrow[\pi_n(g_s)\circ(\pi_n(f_s))^{-1}]{\cong} & \pi_n(V_s, v_s(x)). \end{array}$$

Thus, $\pi_n \circ (\mathbb{V}, x)$ and $\pi_n \circ (\mathbb{W}, y)$ are naturally isomorphic. □

Lemma 6.5. *Let $\mathbb{V}$ and $\mathbb{W}$ be two $\mathbb{R}_+$-spaces. Then, for any $x \in \lim \mathbb{V}$ and $y \in \lim \mathbb{W}$,*

$$d_{\mathrm{I}}\left(\pi_n \circ (\mathbb{V}, x), \pi_n \circ (\mathbb{W}, y)\right) \leqslant d_{\mathrm{HI}}\left((\mathbb{V}, x), (\mathbb{W}, y)\right).$$

Proof. Suppose $(\mathbb{V}, x)$ and $(\mathbb{W}, y)$ are δ-homotopy-interleaved for some $\delta \geqslant 0$. Then there exist $\mathbb{V}', \mathbb{W}' : (\mathbb{R}_+, \leqslant) \to \mathbf{Top}^*$ such that $(\mathbb{V}', x') \simeq (\mathbb{V}, x)$, $(\mathbb{W}', y') \simeq (\mathbb{W}, y)$ for some $x' \in \lim \mathbb{V}'$ and $y' \in \lim \mathbb{W}'$, and $(\mathbb{V}', x')$ and $(\mathbb{W}', y')$ are δ-interleaved. By Lemma 6.4, we know that $\pi_n \circ (\mathbb{V}, x)$ and $\pi_n \circ (\mathbb{V}', x')$ are 0-interleaved, and $\pi_n \circ (\mathbb{W}, y)$ and $\pi_n \circ (\mathbb{W}', y')$ are 0-interleaved. Thus,

$$\begin{aligned}
d_{\mathrm{I}}\left(\pi_n \circ (\mathbb{V}, x),\, \pi_n \circ (\mathbb{W}, y)\right) &\leqslant d_{\mathrm{I}}\left(\pi_n \circ (\mathbb{V}, x),\, \pi_n \circ (\mathbb{V}', x')\right) + d_{\mathrm{I}}\left(\pi_n \circ (\mathbb{V}', x'),\, \pi_n \circ (\mathbb{W}', y')\right) \\
&\quad + d_{\mathrm{I}}\left(\pi_n \circ (\mathbb{W}', y'),\, \pi_n \circ (\mathbb{W}, y)\right) \\
&\leqslant 0 + d_{\mathrm{I}}\left((\mathbb{V}', x'),\, (\mathbb{W}', y')\right) + 0 \\
&\leqslant \delta.
\end{aligned}$$

□

Proof of Theorem 6.1. We first notice that the proof of Theorem 3.12 (see [BL17, Page 20]) can be modified to show that

$$d_{\mathrm{HI}}\left(\left(\left|\mathrm{VR}(X)\right|, x_0\right),\, \left(\left|\mathrm{VR}(Y)\right|, y_0\right)\right) \leqslant 2 \cdot d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

Applying Lemma 6.5, for each $n \in \mathbb{Z}_{\geqslant 1}$ we obtain

$$\begin{aligned}
d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{VR}}(X, x_0), \mathrm{P}\Pi_n^{\mathrm{VR}}(Y, y_0)\right) &= d_{\mathrm{I}}\left(\pi_n \circ \left(\left|\mathrm{VR}(X)\right|, x_0\right),\, \pi_n \circ \left(\left|\mathrm{VR}(Y)\right|, y_0\right)\right) \\
&\leqslant d_{\mathrm{HI}}\left(\left(\left|\mathrm{VR}(X)\right|, x_0\right),\, \left(\left|\mathrm{VR}(Y)\right|, y_0\right)\right) \\
&\leqslant 2 \cdot d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).
\end{aligned}$$

It follows from Corollary 5.11 that

$$d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{K}}(X, x_0), \mathrm{P}\Pi_n^{\mathrm{K}}(Y, y_0)\right) = \tfrac{1}{2} \cdot d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{VR}}(X, x_0), \mathrm{P}\Pi_n^{\mathrm{VR}}(Y, y_0)\right) \leqslant d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

When X and Y are chain-connected, we can take the infimum over all pairs $(x_0, y_0) \in X \times Y$ of both sides of the above Equation to get

$$d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{K}}(X), \mathrm{P}\Pi_n^{\mathrm{K}}(Y)\right) = \tfrac{1}{2} \cdot d_{\mathrm{I}}\left(\mathrm{P}\Pi_n^{\mathrm{VR}}(X), \mathrm{P}\Pi_n^{\mathrm{VR}}(Y)\right) \leqslant d_{\mathrm{GH}}(X, Y).$$

□

Proof of Theorem 6.2. By Theorem 5.18, we have the leftmost inequality below:

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leqslant d_{\mathrm{I}}(\theta_{\pi_1(X)}, \theta_{\pi_1(Y)}) \leqslant d_{\mathrm{I}}(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y)). \quad (9)$$

The second inequality is true because any δ-interleaving between $\mathrm{P}\Pi_1(X)$ and $\mathrm{P}\Pi_1(Y)$ clearly induces a δ-interleaving between $\theta_{\pi_1(X)}$ and $\theta_{\pi_1(Y)}$. It follows from Theorem 5.12 that

$$d_{\mathrm{I}}(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y)) = 2 \cdot d_{\mathrm{I}}\left(\mathrm{P}\Pi_1^{\mathrm{K}}(X), \mathrm{P}\Pi_1^{\mathrm{K}}(Y)\right). \quad (10)$$

Finally we combine Equation (9) and Equation (10) together, and then apply Lemma 6.5 for the case $n = 1$, to obtain

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leqslant 2 \cdot d_{\mathrm{GH}}(X, Y).$$

□

To see that persistent homotopy sometimes provides better approximation of the Gromov-Hausdorff distance than persistent homology, we examine the following example.

Example 6.6 ($\mathbb{S}^1 \times \mathbb{S}^1$ vs. $\mathbb{S}^1 \vee \mathbb{S}^1$)**.** Let $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ denote the torus constructed as the ℓ^∞ product of two unit circles, and let $C = \mathbb{S}^1 \vee \mathbb{S}^1$ be the wedge sum of two unit circles. By Example 5.13 and Example 5.29, the persistent homology groups of C are: for all $k \geqslant 1$,

$$\mathrm{PH}_n^{\mathrm{VR}}(C) \cong \begin{cases} (\mathbb{Z} \times \mathbb{Z}) \left(\frac{k-1}{2k-1}\, 2\pi,\ \frac{k}{2k+1}\, 2\pi\right), & \text{if } n = 2k-1, \\ (\mathbb{Z}^{\times \mathfrak{c}} \times \mathbb{Z}^{\times \mathfrak{c}}) \left[\frac{k}{2k+1}\, 2\pi,\ \frac{k}{2k+1}\, 2\pi\right], & \text{if } n = 2k. \end{cases}$$

Following from the above, we can compute the persistent homology with coefficients in $\mathbb{R}$ and obtain the undecorated persistence diagrams of C as follows: for all $k \geqslant 0$,

$$\begin{aligned} \mathrm{dgm}_{2k+1}(C) &= \left\{ \left(\tfrac{k}{2k+1}\, 2\pi,\ \tfrac{k+1}{2k+3}\, 2\pi\right), \left(\tfrac{k}{2k+1}\, 2\pi,\ \tfrac{k+1}{2k+3}\, 2\pi\right) \right\}, \\ \mathrm{dgm}_{2k+2}(C) &= \varnothing. \end{aligned}$$

It is clear that $\mathrm{dgm}_0(C) = \mathrm{dgm}_0(T^2) = \{(0,\infty)\}$. Together with Example 5.25, for all $k \geqslant 0$, we have $\mathrm{dgm}_{2k+1}(T^2) = \mathrm{dgm}_{2k+1}(C)$, $\mathrm{dgm}_{4k+4}(T^2) = \mathrm{dgm}_{4k+4}(C)$,

$$\mathrm{dgm}_{4k+2}(T^2) = \left\{ \left(\tfrac{k}{2k+1}\, 2\pi,\ \tfrac{k+1}{2k+3}\, 2\pi\right) \right\} \text{ and } \mathrm{dgm}_{4k+2}(C) = \varnothing.$$

Thus,

$$\begin{aligned} \sup_{n \geqslant 0} d_{\mathrm{B}}\left(\mathrm{dgm}_n(T^2), \mathrm{dgm}_n(C)\right) &= \sup_{n=4k+2, k\geqslant 0} d_{\mathrm{B}}\left(\mathrm{dgm}_n(T^2), \mathrm{dgm}_n(C)\right) \\ &= \sup_{k \geqslant 0} d_{\mathrm{B}}\left(\left\{\left(\tfrac{k}{2k+1}\, 2\pi,\ \tfrac{k+1}{2k+3}\, 2\pi\right)\right\}, \varnothing\right) \\ &= \sup_{k\geqslant 0} \tfrac{\pi}{(2k+3)(2k+1)} = \tfrac{\pi}{3}. \end{aligned}$$

Thus, the best approximation of $d_{\mathrm{GH}}(T^2, C)$ given by persistence diagrams is obtained at homological dimension 2. Now let us see that the same lower bound for $d_{\mathrm{GH}}(T^2, C)$ can already be realized by the interleaving distance between persistent fundamental groups. Recall that

$$\mathrm{P\Pi}_1(T^2) = (\mathbb{Z} \times \mathbb{Z})\left(0, \tfrac{2\pi}{3}\right) \text{ and } \mathrm{P\Pi}_1(C) = (\mathbb{Z} * \mathbb{Z})\left(0, \tfrac{2\pi}{3}\right).$$

We claim that

$$d_{\mathrm{I}}(\mathrm{P\Pi}_1(T^2), \mathrm{P\Pi}_1(C)) = \tfrac{\pi}{3}.$$

Indeed, because the composition of group homomorphisms $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ can never be $\mathrm{Id}_{\mathbb{Z}*\mathbb{Z}}$, it must be true that $d_{\mathrm{I}}(\mathrm{P\Pi}_1(T^2), \mathrm{P\Pi}_1(C)) \geqslant \frac{\pi}{3}$. By directly checking that $\mathrm{P\Pi}_1(T^2)$ and $\mathrm{P\Pi}_1(C)$ are $\frac{\pi}{3}$-interleaved, we can conclude that $d_{\mathrm{I}}(\mathrm{P\Pi}_1(T^2), \mathrm{P\Pi}_1(C)) = \frac{\pi}{3}$.

Example 6.7 ($\mathbb{S}^1 \times \mathbb{S}^m$ vs. $\mathbb{S}^1 \vee \mathbb{S}^m$)**.** For $m \geqslant 1$, let $\mathbb{S}^m$ denote the m-sphere of radius 1, equipped with the geodesic distance. We denote $a_m := \frac{1}{2}\arccos(-\frac{1}{m+1})$. By the remarks on Page 508 of [Kat83], $(\mathbb{S}^m)^\epsilon$ is homotopy equivalent to $\mathbb{S}^m$ for any $\epsilon \in (0, a_m)$ and the inclusion $(\mathbb{S}^m)^\epsilon \hookrightarrow (\mathbb{S}^m)^{\epsilon'}$ is a homotopy equivalence for $0 < \epsilon \leqslant \epsilon' \leqslant a_m$. Thus,

$$(|\mathrm{VR}_{2\bullet}(\mathbb{S}^m)|)\,|_{(0,a_m)} \cong (\mathbb{S}^m)^\bullet|_{(0,a_m)} \cong \mathbb{S}^m(0, a_m),$$

where $|_{(0,a_m)}$ means restriction of functors to the interval $(0, a_m)$ and $\mathbb{S}^m(0, a_m)$ is an interval persistence module (see Page 13). Since $a_m \leqslant a_1$, we have $(|\mathrm{VR}_{2\bullet}(\mathbb{S}^1)|)\,|_{(0,a_m)} \cong \mathbb{S}^1(0, a_m)$, and thus, for each $n \geqslant 1$,

	$\mathrm{P\Pi}_n^{\mathrm{VR}}\|_{(0,2a_m)}$	$\mathrm{PH}_n^{\mathrm{VR}}\|_{(0,2a_m)}$
$\mathbb{S}^1\times\mathbb{S}^m$	$(\pi_n(\mathbb{S}^1\times\mathbb{S}^m))\,(0,2a_m)$	$(\mathrm{H}_n(\mathbb{S}^1\times\mathbb{S}^m))\,(0,2a_m)$
$\mathbb{S}^1\vee\mathbb{S}^m$	$(\pi_n(\mathbb{S}^1\vee\mathbb{S}^m))\,(0,2a_m)$	$(\mathrm{H}_n(\mathbb{S}^1\vee\mathbb{S}^m))\,(0,2a_m)$

When $n=m\geqslant 2$, we apply Proposition 3.1 to the above table to obtain

	$\mathrm{P\Pi}_m^{\mathrm{VR}}\|_{(0,2a_m)}$	$\mathrm{PH}_m^{\mathrm{VR}}\|_{(0,2a_m)}$
$\mathbb{S}^1\times\mathbb{S}^m$	$\mathbb{Z}(0,2a_m)$	$\mathbb{Z}(0,2a_m)$
$\mathbb{S}^1\vee\mathbb{S}^m$	$(\mathbb{Z}[t,t^{-1}])(0,2a_m)$	$\mathbb{Z}(0,2a_m)$

Because the composition $\mathbb{Z}[t,t^{-1}]\to\mathbb{Z}\to\mathbb{Z}[t,t^{-1}]$ can never be $\mathrm{Id}_{\mathbb{Z}[t,t^{-1}]}$, the leftmost inequality below is true:

$$\frac{1}{2}\cdot a_m\leqslant\frac{1}{2}\cdot d_{\mathrm{I}}\left(\mathrm{P\Pi}_m^{\mathrm{VR}}\left(\mathbb{S}^1\vee\mathbb{S}^m\right),\mathrm{P\Pi}_m^{\mathrm{VR}}\left(\mathbb{S}^1\times\mathbb{S}^m\right)\right)\leqslant d_{\mathrm{GH}}\left(\mathbb{S}^1\vee\mathbb{S}^m,\mathbb{S}^1\times\mathbb{S}^m\right).$$

The rightmost inequality follows from Theorem 6.1. On the contrary, the restriction of persistent homology $\mathrm{PH}_m^{\mathrm{VR}}|_{(0,2a_m)}$ is not enough to distinguish $\mathbb{S}^1\vee\mathbb{S}^m$ and $\mathbb{S}^1\times\mathbb{S}^m$. The reason to work on the restriction instead of the whole persistent group is that homotopy groups $\pi_n\left(|\mathrm{VR}_\epsilon(\mathbb{S}^m)|\right)$ (or homology groups $\mathrm{H}_n\left(|\mathrm{VR}_\epsilon(\mathbb{S}^m)|\right)$) are not totally known when $\epsilon\geqslant a_m$.

When $n=1$, we have seen in Example 6.6 that

	$\mathrm{P\Pi}_1^{\mathrm{VR}}$	$\mathrm{PH}_1^{\mathrm{VR}}$
$\mathbb{S}^1\times\mathbb{S}^1$	$(\mathbb{Z}\times\mathbb{Z})(0,\frac{2\pi}{3})$	$(\mathbb{Z}\times\mathbb{Z})(0,\frac{2\pi}{3})$
$\mathbb{S}^1\vee\mathbb{S}^1$	$(\mathbb{Z}*\mathbb{Z})(0,\frac{2\pi}{3})$	$(\mathbb{Z}\times\mathbb{Z})(0,\frac{2\pi}{3})$

Example 6.8 ($\mathbb{S}^1\times\mathbb{S}^1$ vs. $\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2$)**.** By an argument similar to that of Example 6.7, we obtain the following table for $a_2=\frac{1}{2}\arccos(-\frac{1}{3})$ and each $n\geqslant 1$:

	$\mathrm{P\Pi}_n^{\mathrm{VR}}\|_{(0,2a_2)}$	$\mathrm{PH}_n^{\mathrm{VR}}\|_{(0,2a_2)}$
$\mathbb{S}^1\times\mathbb{S}^1$	$(\pi_n(\mathbb{S}^1\times\mathbb{S}^1))\,(0,2a_2)$	$(\mathrm{H}_n(\mathbb{S}^1\times\mathbb{S}^1))\,(0,2a_2)$
$\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2$	$(\pi_n(\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2))\,(0,2a_2)$	$(\mathrm{H}_n(\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2))\,(0,2a_2)$

Because $\mathbb{S}^1\times\mathbb{S}^1$ and $\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2$ have isomorphic homology groups in all dimensions, their persistent homology restricted on the interval $(0,2a_2)$ are also isomorphic. However,

$$\mathrm{P\Pi}_1^{\mathrm{VR}}|_{(0,2a_2)}(\mathbb{S}^1\times\mathbb{S}^1)=(\mathbb{Z}\times\mathbb{Z})(0,2a_2)\ \text{and}\ \mathrm{P\Pi}_1^{\mathrm{VR}}|_{(0,2a_2)}(\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2)=(\mathbb{Z}*\mathbb{Z})(0,2a_2)$$

are not isomorphic. Moreover, because the composition of group homomorphisms $\mathbb{Z}*\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}*\mathbb{Z}$ can never be $\mathrm{Id}_{\mathbb{Z}*\mathbb{Z}}$, the leftmost inequality below is true:

$$\frac{1}{2}\cdot a_2\leqslant\frac{1}{2}\cdot d_{\mathrm{I}}\left(\mathrm{P\Pi}_1^{\mathrm{VR}}\left(\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2\right),\mathrm{P\Pi}_1^{\mathrm{VR}}\left(\mathbb{S}^1\times\mathbb{S}^1\right)\right)\leqslant d_{\mathrm{GH}}\left(\mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2,\mathbb{S}^1\times\mathbb{S}^1\right).$$

The rightmost inequality follows from Theorem 6.1 with $n=1$.

6.2. **Second Proof of Stability for** $\mathrm{P\Pi}_1(\bullet)$**.** In this section, inspired by Wilkins' work [Wil11] we provide an alternative proof of Theorem 6.3. This alternative proof is more constructive and independent of the notion of persistent K-fundamental groups. Let us recall the theorem here:

Theorem 6.3 (d_{I}-stability for $\mathrm{P\Pi}_1(\bullet)$)**.** *Let (X, x_0) and (Y, y_0) be pointed compact metric spaces. Then*

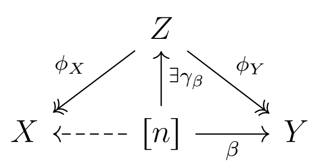
$$d_{\mathrm{I}}(\mathrm{P\Pi}_1(X, x_0), \mathrm{P\Pi}_1(Y, y_0)) \leqslant 2 \cdot d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$

Based on [Wil11, Lemma 6.3.3], we first introduce a way to construct maps between discrete fundamental groups and prove Lemma 6.10.

Definition 6.9 ((ϵ, δ)-homomorphisms)**.** Let X and Y be compact metric spaces with basepoints x_0 and y_0, respectively. Let

$$R : X \overset{\phi_X}{\twoheadleftarrow} Z \overset{\phi_Y}{\twoheadrightarrow} Y$$

be a pointed tripod (i.e., there exits $z_0 \in Z$ such that $\phi_X(z_0) = x_0$ and $\phi_Y(z_0) = y_0$) with $\mathrm{dis}(R) \leqslant \epsilon$. Since ϕ_Y is surjective, for each $\delta > 0$ and each δ-loop $\beta : [n] \to Y$, there is a discrete loop γ_β in Z such that $\phi_Y \circ \gamma_\beta = \beta$ and $\gamma_\beta(0) = \gamma_\beta(n) = z_0$, as expressed by the following diagram:



Using the above, we define a map

$$\psi_\delta^{\delta+\epsilon} : \pi_1^\delta(Y, y_0) \to \pi_1^{\epsilon+\delta}(X, x_0) \text{ such that } [\beta]_\delta \mapsto [\phi_Y \circ \gamma_\beta]_{\epsilon+\delta},$$

and similarly a map

$$\phi_\delta^{\delta+\epsilon} : \pi_1^\delta(Y, y_0) \to \pi_1^{\epsilon+\delta}(X, x_0) \text{ such that } [\alpha]_\delta \mapsto [\phi_X \circ \gamma_\alpha]_{\epsilon+\delta}.$$

We call the maps $\psi_\delta^{\delta+\epsilon}$ and $\phi_\delta^{\delta+\epsilon}$ the *(ϵ, δ)-homomorphisms induced by the tripod R.*

In [Wil11], Wilkins' construction of group homomorphisms between discrete fundamental groups is induced by some isometric embeddings of X and Y into a common metric space Z, and is applicable to any $\epsilon > 2d_{\mathrm{H}}^Z(X, Y)$. Notice that given a tripod R between metric spaces X and Y, R induces a metric space Z_R which X and Y are isometrically embedded into, with $d_{\mathrm{H}}^{Z_R}(X, Y) = 2\,\mathrm{dis}(R)$. We will prove in the next lemma that Definition 6.9 defines group homomorphisms for all $\epsilon > 2\,\mathrm{dis}(R) = d_{\mathrm{H}}^{Z_R}(X, Y)$, which is an improvement over Wilkins' construction.

Lemma 6.10. *With the notation from Definition 6.9, the (ϵ, δ)-homomorphisms induced by a tripod R:*

$$\psi_\delta^{\delta+\epsilon} : \pi_1^\delta(Y, y_0) \to \pi_1^{\delta+\epsilon}(X, x_0) \text{ and } \phi_\delta^{\delta+\epsilon} : \pi_1^\delta(X, x_0) \to \pi_1^{\delta+\epsilon}(Y, y_0)$$

are well-defined group homomorphisms. Furthermore,

$$\psi_\epsilon := \{\psi_\delta^{\delta+\epsilon}\}_{\delta>0} \in \mathrm{Hom}^\epsilon(\mathrm{P\Pi}_1(Y), \mathrm{P\Pi}_1(X)),\ \phi_\epsilon := \{\psi_\delta^{\delta+\epsilon}\}_{\delta>0} \in \mathrm{Hom}^\epsilon(\mathrm{P\Pi}_1(X), \mathrm{P\Pi}_1(Y)),$$

and

$$\psi_\epsilon \circ \phi_\epsilon = 1_{\mathrm{P\Pi}_1(X)}^{2\epsilon},\ \phi_\epsilon \circ \psi_\epsilon = 1_{\mathrm{P\Pi}_1(Y)}^{2\epsilon}.$$

In other words, $\mathrm{P\Pi}_1(X)$ *and* $\mathrm{P\Pi}_1(Y)$ *are ϵ-interleaved via ψ_ϵ and ϕ_ϵ.*

Proof. Fix $\delta > 0$. For a δ-loop β in Y, since ϕ_Y is surjective, let γ_β be a discrete loop in Z such that $\phi_Y(\gamma_\beta) = \beta$ and $\gamma_\beta(0) = \gamma_\beta(n) = z_0$. Because of the fact that $(\phi_X(\gamma_\beta), \beta) \in R_{\mathcal{L}}$ and by Lemma 4.22, $\phi_X(\gamma_\beta)$ is an $(\epsilon + \delta)$-loop in X. So $\psi_\delta^{\delta+\epsilon}$ maps to $\pi_1^{\epsilon+\delta}(X, x_0)$.

Claim 1: $\psi_\delta^{\delta+\epsilon}$ is independent of the choice of γ_β.

Suppose $\beta = y_0 y_1 \cdots y_n$ with $y_n = y_0$ is a δ-loop in Y. Suppose that $\gamma_\beta = z_0 \cdots z_n$ and $\gamma'_\beta = z'_0 \cdots z'_n$, where $z_n = z'_n = z'_0 = z_0$ the basepoint of Z. For each $i \in [n]$, as $y_i = \phi_Y(z_i) = \phi_Y(z'_i)$, we have

$$d_X(\phi_X(z_i), \phi_X(z'_i)) \leqslant d_Y(y_i, y_i) + \operatorname{dis}(R) \leqslant \epsilon,$$

and for each $i = 0, \cdots, n-1$, we have

$$d_X(\phi_X(z_i), \phi_X(z'_{i+1})) \leqslant d_Y(y_i, y_{i+1}) + \operatorname{dis}(R) \leqslant \delta + \epsilon.$$

Thus, inserting $\phi_X(z'_i)$ between $\phi_X(z_{i-1})$ and $\phi_X(z_i)$ is a basic move up to $(\delta+\epsilon)$-homotopy, for all $i = 1, \cdots, n$. This results into the following $(\delta + \epsilon)$-chain in X:

$$\phi_X(z_0)\phi_X(z'_1)\phi_X(z_1)\cdots\phi_X(z'_n)\phi_X(z_n).$$

By sequentially removing $\phi_X(z_i)$ for each $i = 1, \cdots, n$, we obtain a $(\delta + \epsilon)$-homotopy from $\phi_X(\gamma_\beta)$ to $\phi_X(\gamma'_\beta)$. Hence, $[\phi_X(\gamma_\beta)]_{\delta+\epsilon} = [\phi_X(\gamma'_\beta)]_{\delta+\epsilon}$.

Claim 2: $\psi_\delta^{\delta+\epsilon}$ is independent of the choice of β.

Suppose $\beta \sim_1^\delta \beta'$ in Y, where $\beta = y_0 y_1 \cdots y_n$ and $\beta' = y'_0 y'_1 \cdots y'_m$ with $y_n = y'_m = y'_0 = y_0$. Then there is a δ-homotopy in Y:

$$H_Y = \{\beta = \beta_0, \beta_1, \cdots, \beta_{k-1}, \beta_k = \beta'\}$$

such that each β_j differs from β_{j-1} by a basic move. For each β_j, let γ_{β_j} be a discrete loop in Z such that $\phi_Y(\gamma_{\beta_j}) = \beta_j$ and $\gamma_{\beta_j}(0) = \gamma_{\beta_j}(n) = z_0$. Then

$$H_X = \left\{\phi_X(\gamma_{\beta_0}), \phi_X(\gamma_{\beta_1}), \cdots, \phi_X(\gamma_{\beta_{k-1}}), \phi_X(\gamma_{\beta_k})\right\}$$

is an $(\epsilon + \delta)$-homotopy between $\phi_X(\gamma_\beta)$ and $\phi_X(\gamma_{\beta'})$. Therefore,

$$\psi_\delta^{\delta+\epsilon}([\beta]_\delta) = [\phi_X(\gamma_\beta)]_{\epsilon+\delta} = [\phi_X(\gamma_{\beta'})]_{\epsilon+\delta} = \psi_\delta^{\delta+\epsilon}([\beta']_\delta).$$

Claim 3: $\psi_\delta^{\delta+\epsilon}$ is a group homomorphism.

Take two δ-loops β and β' in Y, and choose γ_β and $\gamma_{\beta'}$ so that $\psi_\delta^{\delta+\epsilon}([\beta]_\delta) = [\phi_X(\gamma_\beta)]_{\epsilon+\delta}$ and $\psi_\delta^{\delta+\epsilon}([\beta']_\delta) = [\phi_X(\gamma_{\beta'})]_{\epsilon+\delta}$ as before. It follows from $\phi_X(\gamma_\beta * \gamma_{\beta'}) = \phi_X(\gamma_\beta) * \phi_X(\gamma_{\beta'})$ that $(\phi_X(\gamma_\beta) * \phi_X(\gamma_{\beta'}), \beta * \beta') \in R_{\mathcal{L}}$. Thus,

$$\psi_\delta^{\delta+\epsilon}([\beta * \beta']_\delta) = [\phi_X(\gamma_\beta) * \phi_X(\gamma_{\beta'})]_{\epsilon+\delta} = \psi_\delta^{\delta+\epsilon}([\beta]_\delta)\psi_\delta^{\delta+\epsilon}([\beta']_\delta).$$

Claim 4: $\mathrm{P}\Pi_1(X)$ and $\mathrm{P}\Pi_1(Y)$ are ϵ-interleaved via ψ_ϵ and ϕ_ϵ.

Note that the following diagram commutes for all $\delta \leqslant \delta'$:

$$\begin{array}{ccc} \mathrm{P}\Pi_1^\delta(Y) & \xrightarrow{\psi_\epsilon} & \mathrm{P}\Pi_1^{\delta+\epsilon}(X) \\ {\scriptstyle \Phi^Y_{\delta,\delta'}}\downarrow & & \downarrow{\scriptstyle \Phi^X_{\delta+\epsilon,\delta'+\epsilon}} \\ \mathrm{P}\Pi_1^{\delta'}(Y) & \xrightarrow[\psi_\epsilon]{} & \mathrm{P}\Pi_1^{\delta'+\epsilon}(X). \end{array}$$

Indeed, given a δ-loop β in Y, let γ_β be a discrete loop in Z chosen as before. Then,

$$\psi_\epsilon \circ \Phi^Y_{\delta,\delta'}([\beta]_\delta) = \psi_\epsilon([\beta]_{\delta'}) = [\phi_X(\gamma_\beta)]_{\delta'+\epsilon},$$
$$\Phi^X_{\delta+\epsilon,\delta'+\epsilon} \circ \psi_\epsilon([\beta]_\delta) = \Phi^X_{\delta+\epsilon,\delta'+\epsilon}([\phi_X(\gamma_\beta)]_{\delta+\epsilon}) = [\phi_X(\gamma_\beta)]_{\delta'+\epsilon}.$$

Therefore, $\psi_\epsilon \in \mathrm{Hom}^\epsilon(\mathrm{P}\Pi_1(Y), \mathrm{P}\Pi_1(X))$ and a similar argument can be applied to prove that $\phi_\epsilon \in \mathrm{Hom}^\epsilon(\mathrm{P}\Pi_1(X), \mathrm{P}\Pi_1(Y))$. Also, by directly checking the relevant definitions, it is not hard to see that $\psi_\epsilon \circ \phi_\epsilon = 1^{2\epsilon}_{\mathrm{P}\Pi_1(X)}$ and $\phi_\epsilon \circ \psi_\epsilon = 1^{2\epsilon}_{\mathrm{P}\Pi_1(Y)}$. □

It is clear that Lemma 6.10 immediately implies Theorem 6.3.

The next lemma refines Lemma 6.3.3 from [Wil11] by improving the factors in equations (11) and (12) from $\frac{1}{4}$ to $\frac{1}{2}$:

Lemma 6.11. *Let X and Y be chain-connected compact metric spaces.*

(1) Assume that there exist $\delta > \lambda > 0$ and $\epsilon > 0$ such that $\Phi^X_{\lambda,\epsilon+\delta}$ is surjective. If

$$d_{\mathrm{GH}}(X, Y) < \tfrac{1}{2}\min\{\epsilon, \delta - \lambda\}, \tag{11}$$

then there exists a pointed tripod R such that for any $(x_0, y_0) \in R$, the (ϵ, δ)-homomorphism $\psi^{\delta+\epsilon}_\delta : \pi^\delta_1(Y, y_0) \to \pi^{\delta+\epsilon}_1(X, x_0)$ induced by R is surjective.

(2) Suppose that there exist $\lambda < \epsilon$ such that $\Phi^X_{t_1,t_2}$ and $\Phi^Y_{t_1,t_2}$ are surjective for every $\lambda \leqslant t_1 < t_2 \leqslant \epsilon$. Given $\lambda < \delta < \delta' < \epsilon' < \epsilon$, if

$$d_{\mathrm{GH}}(X, Y) < \tfrac{1}{2}\min\{\delta - \lambda, \delta' - \delta, \epsilon - \epsilon'\}, \tag{12}$$

then we have the following commutative diagram:

$$\begin{array}{ccc} \pi^{\epsilon'}_1(Y) & \xrightarrow{\psi^{\delta'}_\delta} & \pi^\epsilon_1(X) \\ \Phi^Y_{\delta',\epsilon'} \uparrow & & \uparrow \Phi^X_{\delta,\epsilon} \\ \pi^{\delta'}_1(Y) & \xleftarrow[\phi^{\delta'}_\delta]{} & \pi^\delta_1(X) \end{array}$$

Proof. For Part (1), let $\tau := \frac{1}{2}\min\{\epsilon, \delta - \lambda\}$. Since $d_{\mathrm{GH}}(X, Y) < \tau$, there exists a tripod

$$R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$$

with $\mathrm{dis}(R) < 2\tau \leqslant \epsilon$. Take a pair $(x_0, y_0) \in R$, and then let X and Y are based at x_0 and y_0, respectively. Let $\psi^{\delta+\epsilon}_\delta : \pi^\delta_1(Y, y_0) \to \pi^{\delta+\epsilon}_1(X, x_0)$ be the (ϵ, δ)-homomorphism induced by R given in Definition 6.9. It has been proved in Lemma 6.10 that $\psi^{\delta+\epsilon}_\delta$ is a group homomorphism. It remains to show $\psi^{\delta+\epsilon}_\delta$ is surjective. Let α be an $(\epsilon + \delta)$-loop in X. Because $\Phi^X_{\lambda,\epsilon+\lambda}$ is surjective, there exists a λ-loop α' in X that refines α. Let $\gamma_{\alpha'}$ be a discrete loop in (Z, z_0) such that $\phi_X(\gamma_{\alpha'}) = \alpha'$. By Lemma 4.22, $\phi_Y(\gamma_{\alpha'})$ is a $(\lambda + \mathrm{dis}(R))$-loop, and thus a $(\delta + \epsilon)$-loop because $\lambda < \delta$ and $\mathrm{dis}(R) < \epsilon$. It follows that

$$\psi^{\delta+\epsilon}_\delta([\phi_Y(\gamma_{\alpha'})]_\delta) = [\phi_X(\gamma_{\alpha'})]_{\epsilon+\delta} = [\alpha]_{\epsilon+\delta}.$$

Hence, $\psi^{\delta+\epsilon}_\delta$ is surjective. As X and Y are chain-connected, the same result holds for any other choice of basepoints.

For Part (2), let τ be equal to the right side of Equation (12). As in Part (1), we choose a tripod R with $\mathrm{dis}(R) < 2\tau$. Take a pair $(x_0, y_0) \in R$, and then let X and Y are based at

x_0 and y_0, respectively. Let α be a δ-loop in X, and let γ_α be a discrete loop in (Z, z_0) such that $\phi_X(\gamma_\alpha) = \alpha$. Since $\delta + 2\tau \leqslant \delta'$, we have $\phi_\delta^{\delta'}([\alpha]_\delta) = [\phi_Y(\gamma_\alpha)]_{\delta'}$. Thus,

$$\psi_\delta^{\delta'} \circ \Phi^Y_{\delta',\epsilon'} \circ \phi_\delta^{\delta'}([\alpha]_\delta) = \psi_\delta^{\delta'} \circ \Phi^Y_{\delta',\epsilon'}([\phi_Y(\gamma_\alpha)]_{\delta'}) = \psi_\delta^{\delta'}([\phi_Y(\gamma_\alpha)]_{\epsilon'}) = [\alpha]_\epsilon = \Phi^X_{\delta,\epsilon}([\alpha]_\delta).$$

□

7. Application to Finite Metric Spaces

In this section, we analyze several examples of finite metric spaces to further understand the generalized subdendrograms obtained in Lemma 4.27. In general, given a pointed metric space (X, x_0), the set of discrete loops $\mathcal{L}(X, x_0)$ is in bijection with the set $X^{\mathbb{N}} = \{\{x_i\}_{i\in\mathbb{N}} : x_i \in X\}$. Even when X is a finite metric space, $\mathcal{L}(X, x_0)$ can have large cardinality, making the illustration of its subdendrogram rather difficult. To simplify the graphical representation of the resulting subdendrograms, we introduce the equivalence relation $\sim$ on $\mathcal{L}(X, x_0)$ given by:

$$\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ have the } \textit{same} \text{ birth time } \delta \geqslant 0 \text{ and } \gamma \sim_1^\delta \gamma',$$

which implies that $\gamma \sim_1^\epsilon \gamma'$ for all $\epsilon \geqslant \delta$. Clearly, $\sim$ is an equivalence relation and we denote

$$L(X, x_0) := \mathcal{L}(X, x_0) / \sim,$$

and write $[\gamma]$ for the equivalence class of a discrete loop γ under the relation $\sim$. Let $p_X : \mathcal{L}(X, x_0) \twoheadrightarrow L(X, x_0)$ be the resulting quotient map, i.e., $p_X(\gamma) = [\gamma]$. It is not hard to see that $L(X, x_0)$ is a monoid under the operation $p_X(\gamma) \bullet p_X(\gamma') := p_X(\gamma * \gamma')$. The existence of inverse elements is not guaranteed because $\gamma * \gamma^{-1}$, where $\mathrm{birth}(\gamma) = \delta > 0$, is a δ-loop and cannot be equivalent to the 0-loop x_0. Let $L^\epsilon(X, x_0)$ be the set of equivalence classes of discrete loops in X with birth time ϵ, i.e.,

$$L^\epsilon(X, x_0) := \left(\mathcal{L}^\epsilon(X, x_0) - \bigcup_{\delta<\epsilon} \mathcal{L}^\delta(X, x_0) \right) \Big/ \sim .$$

It is clear that $L^\epsilon(X, x_0)$ is a semigroup, i.e., it is closed under the operation $\bullet$ which satisfies the associative property. Furthermore, $L^\epsilon(X, x_0)$ is a sub-semigroup of $L(X, x_0)$.

It can be directly checked that the constructions and results in §4.4 all apply here:

- The pseudo-ultrametric $\mu_X^{(1)}$ on $\mathcal{L}(X, x_0)$ induces a pseudo-ultrametric on $L(X, x_0)$, still denoted by $\mu_X^{(1)}$. In particular, for any two discrete loops γ and γ',
$$\mu_X^{(1)}([\gamma], [\gamma']) := \mu_X^{(1)}(\gamma, \gamma').$$
- Let (X, x_0) and (Y, y_0) be in $\mathcal{M}^{\mathrm{pt}}$. Then each pointed tripod
$$R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$$
induces the following tripod between $L(X, x_0)$ and $L(Y, y_0)$:
$$R_L := L(X, x_0) \xleftarrow{p_X \circ \phi_X} \mathcal{L}(Z, z_0) \xrightarrow{p_Y \circ \phi_Y} L(Y, y_0),$$
with $\mathrm{dis}(R) \leqslant \mathrm{dis}(R_L)$.
- Given (X, x_0) and (Y, y_0) in $\mathcal{M}^{\mathrm{pt}}$,
$$d_{\mathrm{GH}}(L(X, x_0), L(Y, y_0)) \leqslant d_{\mathrm{GH}}^{\mathrm{pt}}((X, x_0), (Y, y_0)).$$
- Let X be a compact geodesic space or a finite metric space. The map $\epsilon \mapsto \pi_1^\epsilon(X, x_0)$ induces a generalized subdendrogram over $L(X, x_0)$, denoted by $\theta^{\mathrm{s}}_{L(X,x_0)}$.

Proposition 7.1. *Given $(X, x_0) \in \mathcal{M}^{\text{pt}}$ and $\epsilon > 0$, if $L^\epsilon(X, x_0)$ is non-empty, then $L^\epsilon(X, x_0)$ is isomorphic to a sub-semigroup of $\pi_1^\epsilon(X, x_0)$.*

Proof. It suffices to show the map $f : L^\epsilon(X, x_0) \to \pi_1^\epsilon(X, x_0)$, with $[\gamma] \to [\gamma]_\epsilon$ is an injective semigroup homomorphism. Given discrete loops γ and γ' with birth time ϵ such that $\gamma \sim \gamma'$, we have $\gamma \sim_1^\epsilon \gamma'$. Thus, f is well-defined. Clearly, f preserve the semigroup operation. It remains to check that f is injective. Indeed, if $f([\gamma]) = f([\gamma'])$ for some γ and γ' with birth time ϵ, then $\gamma \sim_1^\epsilon \gamma'$. Thus, we have $\gamma \sim \gamma'$. □

The following corollary follows immediately:

Corollary 7.2. *Let $(X, x_0) \in \mathcal{M}^{\text{pt}}$ and $\epsilon > 0$. If $\pi_1^\epsilon(X, x_0) = 0$ and $L^\epsilon(X, x_0) \neq \varnothing$, i.e., there exists a discrete loop γ_ϵ in X with birth time ϵ, then*

$$L^\epsilon(X, x_0) = \{[\gamma_\epsilon]\}.$$

Finite Tree Metric Spaces. Recall from Page 10 that a finite tree metric space (T, d_T) is a finite metric space where d_T satisfies the four-point condition. It is proved in Theorem 2.1 of [CCR13, Supplementary material] that the k-th homology group of $\mathrm{VR}_\epsilon(T)$ is 0 for all $k \geqslant 1$. By adapting the proof, we obtain the following theorem applicable to persistent fundamental groups:

Theorem 7.3. *Let T be a finite tree metric space and fix $\epsilon > 0$. For each $x_0 \in T$, we have*

$$\mathrm{P}\Pi_1(T, x_0) = \mathbb{0}.$$

Proof. Recall from Theorem 4.15 that $\pi_1^\epsilon(T, x_0) \cong \pi_1(|\mathrm{VR}_\epsilon(T)|, x_0)$. When $\epsilon \geqslant \mathrm{diam}(T)$, $|\mathrm{VR}_\epsilon(T)|$ is contractible and thus has trivial fundamental group. Let us assume $\epsilon < \mathrm{diam}(T)$ and show that $\pi_1(|\mathrm{VR}_\epsilon(T)|, x_0) = 0$ for all $x_0 \in T$ by induction on the cardinality of T.

It is clear that the statement is true when T contains only one point. Now we suppose that $\pi_1(|\mathrm{VR}_\epsilon(T')|, x_0') = 0$ for all $x_0' \in T'$, whenever T' is a tree metric space with cardinality $< n$. Let T be a tree metric space with n points, based at point x_0. Recall the equivalence relation $\sim_0^\epsilon$ given on Page 17, under which T breaks up into equivalence classes $T = \sqcup_\alpha T_\alpha$. When T is not ϵ-connected, each T_α has cardinality $< n$. By the induction hypothesis, if $x_0 \in T_\beta$ for some β, then $\pi_1(|\mathrm{VR}_\epsilon(T)|, x_0) = \pi_1(|\mathrm{VR}_\epsilon(T_\beta)|, x_0) = 0$.

We suppose next that T is ϵ-connected. Let x_1 and x_2 be two points in T such that $d_T(x_1, x_2) = \mathrm{diam}(T)$. We denote $T_1 := T - \{x_1\}$ and $T_2 := T - \{x_2\}$. It has been shown in Proposition 2.2 and Theorem 2.1 of [CCR13, Supplementary material] that the following are true:

- $|\mathrm{VR}_\epsilon(T)| = |\mathrm{VR}_\epsilon(T_1)| \cup |\mathrm{VR}_\epsilon(T_2)|$, where $|\mathrm{VR}_\epsilon(T_i)|$ is path connected for $i = 1, 2$;
- $|\mathrm{VR}_\epsilon(T_1)| \cap |\mathrm{VR}_\epsilon(T_2)| = |\mathrm{VR}_\epsilon(T - \{x_1, x_2\})|$ is path connected.
- $|\mathrm{VR}_\epsilon(T_1)|$, $|\mathrm{VR}_\epsilon(T_2)|$ and $|\mathrm{VR}_\epsilon(T - \{x_1, x_2\})|$ all have trivial fundamental groups based at x_0, by the induction hypothesis.

Let x_0 be distinct from x_1 and x_2. Then we can apply the van-Kampen theorem (see [Hat01, Theorem 1.20]) to conclude:

$$\pi_1(|\mathrm{VR}_\epsilon(T)|, x_0) = \pi_1(|\mathrm{VR}_\epsilon(T_1)|, x_0) * \pi_1(|\mathrm{VR}_\epsilon(T_2)|, x_0) = 0$$

As $|\mathrm{VR}_\epsilon(T)|$ is connected, we finally obtain $\pi_1(|\mathrm{VR}_\epsilon(T)|, \tilde{x}_0) = \pi_1(|\mathrm{VR}_\epsilon(T)|, x_0)$ for any $\tilde{x}_0 \in T$. □

7.1. **Finite Metric Spaces Arising from Graphs.** Next we study the discrete loop set and the persistent fundamental group of graphs. All our graphs are finite, undirected and simple (i.e., with no self-loops or multiple edges) graphs, with weight 1 on each edge. Given a graph G, we can obtain a metric space by equipping the vertex set with the shortest path distance d_G. For simplicity, the resulting metric space is written as (G, d_G), where the base space shall be understood as the vertex set of G. The *girth* of a graph is the length of its shortest cycle or ∞ for a forest (see [Ada13]).

Proposition 7.4 (Fact 2.1 & Proposition 2.2, [Ada13])**.** *For any connected graph G and $r \geqslant 1$, the map of fundamental groups*

$$\pi_1\left(|\mathrm{VR}_1(G)|\right) \twoheadrightarrow \pi_1\left(|\mathrm{VR}_r(G)|\right)$$

induced by the inclusion $\mathrm{VR}_1(G) \hookrightarrow \mathrm{VR}_r(G)$ is surjective. Furthermore, if $r \geqslant 1$ is such that G is a graph of girth at least $3r+1$, then

$$\pi_1\left(|\mathrm{VR}_{k-1}(G)|\right) \xrightarrow{\cong} \pi_1\left(|\mathrm{VR}_k(G)|\right)$$

for each $2 \leqslant k \leqslant r$.

Let $n \in \mathbb{Z}_{\geqslant 3}$. A *cycle graph* C_n is a graph on n vertices containing a single cycle through all its vertices. A *star graph* S_n is a tree on n vertices where one vertex (called the *center*) has degree $n-1$ and the others have degree 1. Because of the symmetry, the set of discrete loops $\mathcal{L}(C_n, v_0)$ and the persistent fundamental group $\mathrm{P}\Pi_1(C_n, v_0)$ do not depend on choices of the basepoint v_0. As for S_n, we always choose its center to be the basepoint. For convenience, we denote the vertex set $V(C_n) = \{0, 1, \cdots, n-1\}$ such that $d_{C_n}(i, i+1) = 1$, and denote the vertex set $V(S_n) = \{0, 1, \cdots, n-1\}$, with 0 the basepoint in both cases.

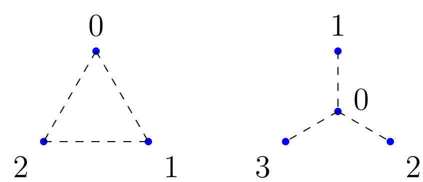


FIGURE 6. Graphs C_3 (left) and S_4 (right).

For later purpose, we introduce another metric space on the set of n points, $E_n := (\{0, 1, \cdots, n-1\}, d_{E_n})$, where

$$d_{E_n}(i,j) = \begin{cases} 1, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Theorem 7.5 (Corollary 6.6, [Ada13])**.** *For any $n \geqslant 3$ and $0 \leqslant r < \frac{n}{2}$, there are homotopy equivalences*

$$|\mathrm{VR}_r(C_n)| \cong \begin{cases} \mathbb{S}^{2l+1}, & \text{if } \frac{l}{2l+1} < \frac{r}{n} < \frac{l+1}{2l+3} \text{ for some } l = 0, 1, \cdots, \\ \bigvee^{n-2r-1} \mathbb{S}^{2l}, & \text{if } \frac{r}{n} = \frac{l}{2l+1} \text{ for some } l = 0, 1, \cdots. \end{cases}$$

For any integer $k \geqslant \frac{1}{2}n$, $|\mathrm{VR}_k(C_n)|$ is contractible.

For $n \in \mathbb{Z}_{\geqslant 3}$ and $r = 0, \cdots, \lfloor \frac{n}{2} \rfloor$, we denote the following r-loop in C_n as

$$\gamma_r := 0r(r+1)\cdots(n-1)0.$$

Although γ_r depends on n, for simplicity of notation we do not specify n unless necessary.

Proposition 7.6. *Fix $n \in \mathbb{Z}_{\geqslant 3}$.*

(1) Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Then

$$\mathrm{P}\Pi_1(C_n) = \mathrm{PH}_1^{\mathrm{VR}}(C_n) = \mathbb{Z}\left[1, \left\lfloor \tfrac{n+2}{3} \right\rfloor\right).$$

Furthermore, for $r = 1, \cdots, \lfloor \frac{n-1}{3} \rfloor$, a generator of $\mathrm{P}\Pi_1^r(C_n)$ is $[\gamma_r]_r$.

(2) We have pseudo-ultrametric space

$$L(C_n) = \left\{[\gamma_0], [\gamma_r^{*k}], [\gamma_s] : r = 1, \cdots, \left\lfloor \tfrac{n-1}{3} \right\rfloor; s = \left\lfloor \tfrac{n+2}{3} \right\rfloor, \cdots, \left\lfloor \tfrac{n}{2} \right\rfloor; k \in \mathbb{Z} - \{0\}\right\},$$

with the pseudo-metric $\mu_{C_n}^{(1)}$ given by: for $k, k' \in \mathbb{Z} - \{0\}$,

$$\mu_{C_n}^{(1)}([\gamma_r^{*k}], [\gamma_{r'}^{*k'}]) = \begin{cases} 0, & \text{if } r = r' = 0 \\ \max\left\{\left\lfloor \tfrac{n-1}{3} \right\rfloor, r\right\}, & \text{if } r' = 0,\ r \geqslant 1 \\ \max\{r, r'\}, & \text{otherwise.} \end{cases}$$

Proof. For Part (1), we first apply Theorem 7.5 to obtain that $|\mathrm{VR}_r(C_n)| \cong \mathbb{S}^1$ iff $1 \leqslant r < \lfloor \frac{n+2}{3} \rfloor$ and $\mathrm{P}\Pi_1^r(C_n) = \mathrm{PH}_1^{\mathrm{VR},r}(C_n) = 0$ otherwise. Clearly, a generator of $\mathrm{P}\Pi_1^1(C_n)$ is $[\gamma_1]_1$. Since the girth of C_n is n, Proposition 7.4 implies that the inclusion $\mathrm{VR}_1(C_n) \hookrightarrow \mathrm{VR}_r(C_n)$ induces an isomorphism

$$\mathrm{P}\Pi_1^1(C_n) \to \mathrm{P}\Pi_1^r(C_n) \text{ with } [\gamma_1]_1 \mapsto [\gamma_1]_r = [\gamma_r]_r,$$

for $r = 1, \cdots, \lfloor \frac{n-1}{3} \rfloor$. Thus, $[\gamma_r]_r$ is a generator of $\mathrm{P}\Pi_1^r(C_n)$ for $r = 1, \cdots, \lfloor \frac{n-1}{3} \rfloor$.

For (2), we first notice that the possible birth times of discrete loops in C_n are $0, \cdots, \lfloor \frac{n}{2} \rfloor$ and each γ_r has birth time r. For $r = 0$ or $\lfloor \frac{n+2}{3} \rfloor, \cdots, \lfloor \frac{n}{2} \rfloor$, because Part (1) implies that $\mathrm{P}\Pi_1^r(C_n) = 0$, by Proposition 7.2 we then have

$$L^r(C_n) = \{[\gamma_r]\}.$$

When $r = 1, \cdots, \lfloor \frac{n+2}{3} \rfloor$, we claim

$$L^r(C_n) = \left\{[\gamma_i^{*k}] : k \in \mathbb{Z} - \{0\}\right\}.$$

It is clear that $\{[\gamma_r^{*k}] : k \in \mathbb{Z} - \{0\}\} \subset L^r(C_n)$, so it remains to show that any discrete loop with birth time r is r-homotopic to γ_r^k for some $k \in \mathbb{Z} - \{0\}$, which is true because $\mathrm{P}\Pi_1^r(C_n)$ is generated by $[\gamma_r]_r$. The calculation of $\mu_{C_n}^{(1)}$ is straightforward, given Proposition 7.4. ☐

Remark 7.7. Note that $\left(L(C_n), \mu_{C_n}^{(1)}\right)$ can be represented by the generalized subdendrogram depicted in Figure 7, where the pseudo-ultrametric induced by the generalized subdendrogram (see Page 17) agrees with $\mu_{C_n}^{(1)}$.

Now let us apply Proposition 7.6 to the cases $n = 3$ and 4. It follows that $\mathrm{P}\Pi_1(C_3) = \mathbb{0}$, and $\left(L(C_3), \mu_{C_3}^{(1)}\right)$ is a two-point pseudo-metric space given by the distance matrix

$$\mu_{C_3}^{(1)} = \begin{array}{c} \begin{array}{cc} [\gamma_0] & [\gamma_1] \end{array} \\ \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \end{array} \begin{array}{c} [\gamma_0] \\ [\gamma_1] \end{array}.$$

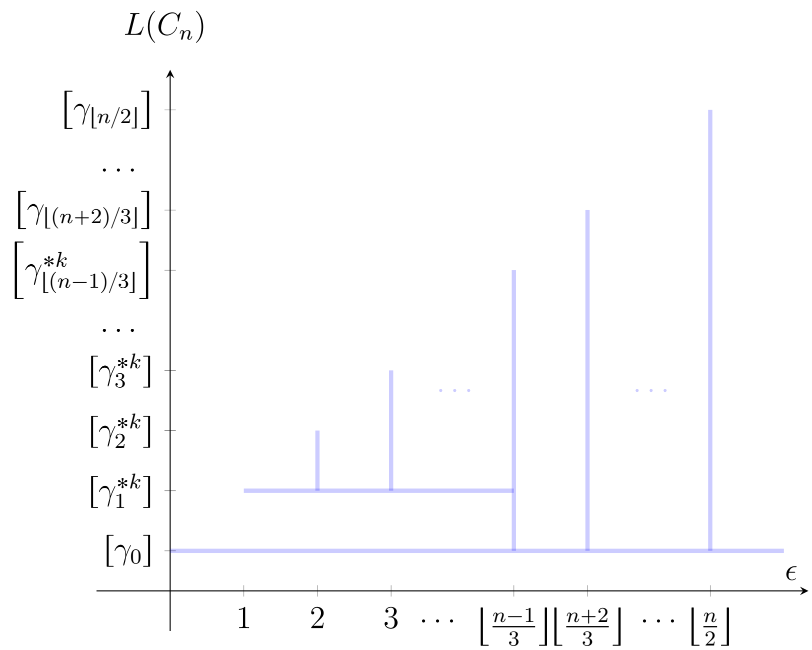


FIGURE 7. Generalized subdendrogram over $L(C_n)$, where k runs over non-zero integers for each $[\gamma_r^{*k}]$ when $r = 1, \cdots, \lfloor \frac{n-1}{3} \rfloor$.

Similarly, we have $P\Pi_1(C_4) = \mathbb{Z}[1, 2)$. The corresponding generalized subdendrograms over $L(C_3)$ and $L(C_4)$ are depicted in Figure 8.

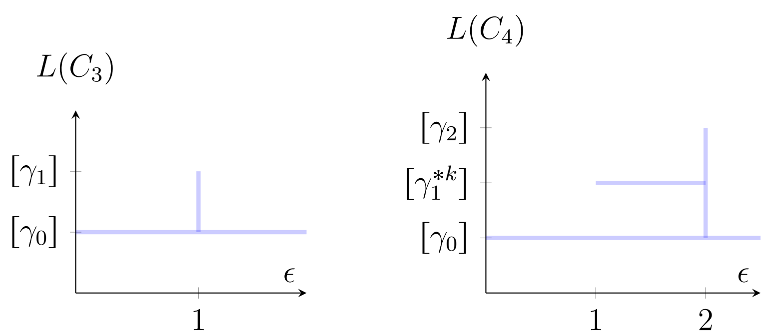


FIGURE 8. Generalized subdendrogram over $L(C_3)$ (left) and $L(C_4)$ (right).

Proposition 7.8. *Fix $n \in \mathbb{Z}_{\geqslant 3}$. Then,*

(1) $P\Pi_1(S_n) = 0$;

(2) $L(S_n) = \{[\lambda_0], [\lambda_1], [\lambda_2]\}$, *where* $\lambda_0 = 0$, $\lambda_1 = 010$ *and* $\lambda_2 = 0120$. *And the pseudo-metric $\mu^{(1)}_{S_n}$ is given by the distance matrix*

$$\mu^{(1)}_{S_n} = \begin{array}{c} \begin{array}{ccc} [\lambda_0] & [\lambda_1] & [\lambda_2] \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 2 \\ & 1 & 2 \\ & & 2 \end{array}\right] \end{array} \begin{array}{c} \\ [\lambda_0] \\ [\lambda_1] \\ [\lambda_2] \end{array}$$

Proof. The possible birth times of discrete loops in S_n are $0, 1$ and 2. Since S_n is a finite tree, by Proposition 7.3 we have $\mathrm{P\Pi}_1^r(S_n) = 0$ for all r. It then follows from Proposition 7.2 that there is only one equivalence class for each possible birth time, which are λ_0, λ_1 and λ_2. □

The corresponding generalized subdengrogram over $L(S_n)$ is depicted in Figure 9.

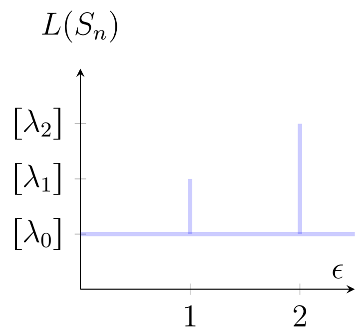


FIGURE 9. Generalized subdendrogram over $L(S_n)$.

Next we compute the Gromov-Hausdorff distance between cycle graphs and star graphs, and then compare it with several distances given in Proposition 7.11.

Proposition 7.9. *For $m, n \in \mathbb{Z}_{\geqslant 3}$, we have*

$$d_{\mathrm{GH}}(C_m, S_n) = \begin{cases} \frac{1}{2}\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right), & \text{if } m \geqslant 6, \\ 1, & \text{if } 3 \leqslant m \leqslant 5 \text{ and } m < n-1, \\ \frac{1}{2} & \text{if } 3 \leqslant m \leqslant 5 \text{ and } m \geqslant n-1. \end{cases}$$

Remark 7.10. Exactly same statement in Proposition 7.9 is true for $d_{\mathrm{GH}}^{\mathrm{pt}}((C_m, 0), (S_n, \mathbf{0}))$ as well, because the proof still holds if we require all tripods to be pointed, i.e., containing $(0, \mathbf{0})$.

Proposition 7.11. *For $m, n \in \mathbb{Z}_{\geqslant 3}$, we have the following:*

(1) $d_{\mathrm{GH}}\left(\left(C_m, \mu_{C_m}^{(0)}\right), \left(S_n, \mu_{S_n}^{(0)}\right)\right) = \frac{1}{2}$ *when $m \neq n$, and 0 otherwise.*

(2) $\frac{1}{2} \cdot d_{\mathrm{I}}(\mathrm{P\Pi}_1(C_m), \mathrm{P\Pi}_1(S_n)) = \frac{1}{2} \cdot d_{\mathrm{B}}(\mathrm{dgm}_1(C_m), \mathrm{dgm}_1(S_n)) = \frac{1}{4} \cdot \left(\left\lfloor \frac{m+2}{3} \right\rfloor - 1\right).$

Remark 7.12. Since Proposition 7.9 implies that $d_{\mathrm{GH}}(C_m, S_n) \to \infty$ as $m \to \infty$, we can see that the value given by Proposition 7.11 (1) is not ideal as a lower bound for $d_{\mathrm{GH}}(C_m, S_n)$. Although strictly less than $d_{\mathrm{GH}}(C_m, S_n)$, Proposition 7.11 (2) increases to infinity as $m \to \infty$.

Proof of Proposition 7.9. The proof is divided into four cases:

(a) $m \geqslant 6$.
(b) $m \geqslant 4$ and $m \geqslant n - 1$.
(c) $m = 4$ or 5, and $m < n - 1$.
(d) $m = 3$.

To distinguish points in C_m and S_n, we will write the vertex set of C_m as $\{0, 1, \cdots, m-1\}$ and the vertex set of S_n as $\{\mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{n-1}\}$. For any two positive integers l and k, by $l \bmod k$ we will mean the remainder of the Euclidean division of l divided by k.

When $m \geqslant 4$, $\operatorname{diam}(C_m) = \lfloor \frac{m}{2} \rfloor \geqslant 1 = \operatorname{rad}(S_n)$ and each point in C_m has an antipode. Thus, we can apply Proposition 3.6 to obtain

$$\frac{1}{2} \cdot \left(\lfloor \tfrac{m}{2} \rfloor - 1 \right) \leqslant d_{\mathrm{GH}}(C_m, S_n) \leqslant \frac{1}{2} \cdot \lfloor \tfrac{m}{2} \rfloor. \tag{13}$$

For case (a): $m = 6$, we claim that $d_{\mathrm{GH}}(C_m, S_n) = \frac{1}{2} \cdot \left(\lfloor \frac{m}{2} \rfloor - 1 \right)$. Because of Equation (13), it remains to construct a tripod R such that $\operatorname{dis}(R) \leqslant \lfloor \frac{m}{2} \rfloor - 1$. In other words, we want the tripod R to satisfy the following condition: $\forall (i_1, \boldsymbol{j_1}), (i_2, \boldsymbol{j_2}) \in R$,

$$|d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2})| < \lfloor \tfrac{m}{2} \rfloor.$$

Given $m \geqslant 6$, it is always true that $\forall (i_1, \boldsymbol{j_1}), (i_2, \boldsymbol{j_2}) \in R$,

$$\lfloor \tfrac{m}{2} \rfloor < -2 \leqslant d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) \leqslant \lfloor \tfrac{m}{2} \rfloor. \tag{14}$$

Since the leftmost inequality of Equation (14) is strict, we have that

$$\begin{aligned} &|d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2})| = \lfloor \tfrac{m}{2} \rfloor, \\ \text{iff } & d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) = \lfloor \tfrac{m}{2} \rfloor, \\ \text{iff } & d_{C_m}(i_1, i_2) = \lfloor \tfrac{m}{2} \rfloor \text{ and } d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) = 0. \end{aligned}$$

Therefore, to construct a tripod R with $\operatorname{dis}(R) \leqslant \lfloor \frac{m}{2} \rfloor - 1$, it suffices to construct a tripod where any pair of antipodes in C_m do not correspond to the same point in S_n. More precisely, R shall satisfy the condition:

$$\forall (i_1, \boldsymbol{j_1}), (i_2, \boldsymbol{j_2}) \in R, \text{ if } d_{C_m}(i_1, i_2) = \lfloor \tfrac{m}{2} \rfloor, \text{ then } \boldsymbol{j_1} \neq \boldsymbol{j_2}. \tag{$*$}$$

Suppose $m = 2k$ for some $k \geqslant 3$. When $m \geqslant n$, we define a tripod R_1 between C_m and S_n by

$$R_1 := \{(i, \boldsymbol{2i} \bmod n), (k + i, (\boldsymbol{2i} + \boldsymbol{1}) \bmod n) : i = 0, \cdots, k-1\}.$$

When $m < n$, we define a tripod R_2 between C_m and S_n by

$$R_2 := \left\{(i \bmod m, \boldsymbol{2i}), ((k+i) \bmod m, \boldsymbol{2i} + \boldsymbol{1}) : i = 0, \cdots, \lfloor \tfrac{n-1}{2} \rfloor \right\}.$$

Next we assume $m = 2k + 1$ for some integer $k \geqslant 3$. When $m \geqslant n$, we define a tripod R_3 between C_m and S_n by

$$R_3 := \{(0, \boldsymbol{0}), (k - i, (\boldsymbol{2i} + \boldsymbol{1}) \bmod n), (m - i, \boldsymbol{2i} \bmod n) : i = 0, \cdots, k-1\}.$$

When $m < n$, we define a tripod R_4 between C_m and S_n by

$$R_4 := \left\{((k - i) \bmod m, \boldsymbol{2i} + \boldsymbol{1}), ((m-i) \bmod m, \boldsymbol{2i}) : i = 0, \cdots, \lfloor \tfrac{n-1}{2} \rfloor \right\}.$$

It can be directly checked that R_1, R_2, R_3 and R_4 are tripods satisfying Condition ($*$).

For case (b): $m \geqslant 4$ and $m \geqslant n - 1$, we claim that $d_{\mathrm{GH}}(C_m, S_n) = \frac{1}{2} \cdot \left(\lfloor \frac{m}{2} \rfloor - 1 \right)$. Let $k := \lfloor \frac{m}{2} \rfloor$. Because of Equation (13), it remains to construct a tripod R such that $\operatorname{dis}(R) \leqslant \lfloor \frac{m}{2} \rfloor - 1$. We will use the same constructions R_1, R_2, R_3 and R_4 as case (a) in corresponding cases. Indeed, when m is even, we use R_1 if $m \geqslant n$ and R_2 if $m = n - 1$; when m is odd, we use R_3 if $m \geqslant n$ and R_4 if $m = n - 1$. Assume $l = 1, 2, 3, 4$. Let i_1 and i_2 be any two points from C_m, and suppose $(i_1, \boldsymbol{j_1})$ are $(i_2, \boldsymbol{j_2})$ in R_l. If $i_2 = i_1 + k$, then it can be checked that

$$0 \leqslant k - 2 \leqslant d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) \leqslant k - 1. \tag{15}$$

If $d_{C_m}(i_1, i_2) \leqslant k - 1$, then

$$-(k-1) \leqslant -1 \leqslant d_{C_m}(i_1, i_2) - d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) \leqslant k - 1. \tag{16}$$

The second inequality of Equation (16) is true because when $i_1 = i_2 \neq 0$, we always have $\boldsymbol{j_1} = \boldsymbol{j_2}$ due to the construction of R_l, and when $i_1 = i_2 = 0$, we have $d_{S_n}(\boldsymbol{j_1}, \boldsymbol{j_2}) \leqslant 1$ (for R_2 and R_4, this argument relies on the condition $m = n-1$). By Equation (15) and Equation (16), we have $\mathrm{dis}(R_l) \leqslant k-1$.

For case (c): $m = 4$ or 5, and $m < n-1$, we claim that $d_{\mathrm{GH}}(C_m, S_n) = \frac{1}{2} \cdot \left\lfloor \frac{m}{2} \right\rfloor = 1$. By the upper bound from Equation (13), it suffices to show that for each tripod R between C_m and S_n, $\mathrm{dis}(R) \geqslant \left\lfloor \frac{m}{2} \right\rfloor = 2$. By the pigeonhole principle, since $m < n-1$, there must be two elements $\boldsymbol{i}, \boldsymbol{j} \in \{\boldsymbol{1}, \cdots, \boldsymbol{n-1}\}$ such that $(l, \boldsymbol{i}), (l, \boldsymbol{j}) \in R$ for some $l \in C_m$. Therefore, $\mathrm{dis}(R) \geqslant d_{S_n}(\boldsymbol{i}, \boldsymbol{j}) = 2$.

For case (d): $m = 3$, since $\mathrm{diam}(C_3) = 1$ and $\mathrm{diam}(S_n) = 2$, Proposition 3.5 implies that

$$\tfrac{1}{2} = \tfrac{1}{2} \cdot |1-2| \leqslant d_{\mathrm{GH}}(C_m, S_n) \leqslant \tfrac{1}{2} \cdot \max\{1, 2\} = 1.$$

If $m < n-1$, we can apply the pigeonhole principle as in case (c), to show each tripod between C_m and S_n has distortion 2. Thus, $d_{\mathrm{GH}}(C_m, S_n)$ reaches the upper bound value 1. If $m \geqslant n-1$, i.e., $n = 3$ or 4, we claim that $d_{\mathrm{GH}}(C_m, S_n) = \frac{1}{2}$ by constructing tripod with distortion 1. Indeed, we can again utilize the construction R_3 for $n = 3$ and R_4 for $n = 4$. It is not hard to verify that their distortions are both 1. □

Proof of Proposition 7.11. For (1), we first notice that $\left(C_m, \mu^{(0)}_{C_m}\right) \cong (E_m, d_{E_m})$ and $\left(S_n, \mu^{(0)}_{S_n})\right) \cong (E_n, d_{E_n})$, where the metric space (E_n, d_{E_n}) is given in Page 50. When $m = n$, it is clear that $d_{\mathrm{GH}}\left((E_n, d_{E_n}), (E_n, d_{E_n})\right) = 0$. When $m \neq n$, suppose without loss of generality that $m > n$. We claim that

$$d_{\mathrm{GH}}\left((E_m, d_{E_m}), (E_n, d_{E_n})\right) = \tfrac{1}{2}.$$

By Proposition 3.5, $d_{\mathrm{GH}}\left((E_m, d_{E_m}), (E_n, d_{E_n})\right) \leqslant \frac{1}{2}$. It remains to show that $\frac{1}{2}$ is also a lower bound for $d_{\mathrm{GH}}\left((E_m, d_{E_m}), (E_n, d_{E_n})\right)$. Given any pointed tripod R, as E_m has more points than E_n, some distinct points in E_m must correspond to the same point in E_n. Thus, $\mathrm{dis}(R) \geqslant 1 - 0 = 1$.

For Part (2), note that

$$d_{\mathrm{I}}(\mathrm{P\Pi}_1(C_m), \mathrm{P\Pi}_1(S_n)) = d_{\mathrm{I}}(\mathrm{PH}_1^{\mathrm{VR}}(C_m), \mathrm{PH}_1^{\mathrm{VR}}(S_n)) = d_{\mathrm{B}}(\mathrm{dgm}_1(C_m), \mathrm{dgm}_1(S_n)).$$

It follows from Proposition 7.6 (1) that $\mathrm{dgm}_1(C_m) = \left\{\left(1, \left\lfloor \frac{m+2}{3} \right\rfloor\right)\right\}$ for $m > 3$ and $\varnothing$ for $m = 3$, and from Proposition 7.8 (1) it follows that $\mathrm{dgm}_1(S_n) = \varnothing$. Thus,

$$d_{\mathrm{B}}(\mathrm{dgm}_1(C_m), \mathrm{dgm}_1(S_n)) = \tfrac{1}{2} \cdot \left(\left\lfloor \tfrac{m+2}{3} \right\rfloor - 1\right).$$

□

7.2. **Discretization of $\mathbb{S}^1$.** Let the unit circle $\mathbb{S}^1$ be embedded into the complex plane with the center at 0, i.e., $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Again, we equip $\mathbb{S}^1$ with its geodesic metric d. For $n \in \mathbb{Z}_{\geqslant 0}$, we consider the metric subspace of $(\mathbb{S}^1, d)$:

$$\Delta_n := \left\{ e^{\frac{2\pi i k}{n}} : k \in [n-1] \right\},$$

where $i = \sqrt{-1}$. For simplicity, we write $\Delta_n = \{0, 1, \cdots, n-1\}$ and assume 0 is the basepoint. Also, we denote $d|_{\Delta_n \times \Delta_n}$ by d_n and note that $(\Delta_n, d_n) \cong \left(C_n, \frac{2\pi}{n} d_{C_n}\right)$. For example, Figure 10 shows the graphs corresponding to Δ_3 and Δ_4.

Proposition 7.13. *We have the following:*

(1) $d_{\mathrm{GH}}((\Delta_3, d_3), (\Delta_4, d_4)) = \frac{\pi}{4}$.

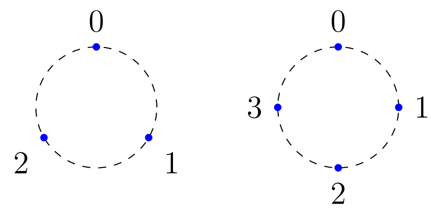


FIGURE 10. Graphs of Δ_3 (left) and Δ_4 (right).

(2) $d_{\mathrm{GH}}\left(\left(\Delta_3, \mu^{(0)}_{\Delta_3}\right), \left(\Delta_4, \mu^{(0)}_{\Delta_4}\right)\right) = \frac{\pi}{4}$.
(3) $d_{\mathrm{GH}}\left(\left(L(\Delta_3), \mu^{(1)}_{\Delta_3}\right), \left(L(\Delta_4), \mu^{(1)}_{\Delta_4}\right)\right) = \frac{\pi}{6}$.
(4) $\frac{1}{2} \cdot d_{\mathrm{I}}(\mathrm{P\Pi}_1(\Delta_3), \mathrm{P\Pi}_1(\Delta_4)) = \frac{1}{2} \cdot d_{\mathrm{B}}(\mathrm{dgm}_1(\Delta_3), \mathrm{dgm}_1(\Delta_4)) = \frac{\pi}{8}$.

Proof. For (1), we first note that $\mathrm{Im}(d_3) = \left\{0, \frac{2\pi}{3}\right\}$ and $\mathrm{Im}(d_4) = \left\{0, \frac{\pi}{2}, \pi\right\}$. For any pointed tripod $R : (\Delta_3, 0) \xleftarrow{\phi} Z \xrightarrow{\psi} (\Delta_4, 0)$, we have

$$\frac{\pi}{2} = \min\left\{\pi - 0, \frac{\pi}{2} - 0\right\} \leqslant \max_{z,z' \in Z} |d_3(\phi(z), \phi(z')) - d_4(\psi(z), \psi(z'))| \leqslant \mathrm{dis}(R),$$

since Δ_4 has more elements than Δ_3. It follows that $d_{\mathrm{GH}}(\Delta_3, \Delta_4) \geqslant \frac{\pi}{4}$. Next we construct a tripod R whose distortion is $\frac{\pi}{2}$, by first defining two set maps f and g as in Figure 11. We set $Z = \Delta_3 \sqcup \Delta_4$, and define set maps $\phi = (\mathrm{Id}_{\Delta_3}, f) : \Delta_3 \sqcup \Delta_4 \to \Delta_3$ and $\psi = (g, \mathrm{Id}_{\Delta_4}) : \Delta_3 \sqcup \Delta_4 \to \Delta_4$. This forms a pointed tripod R. By direct calculation, we obtain $\mathrm{dis}(R) = \frac{\pi}{2}$.

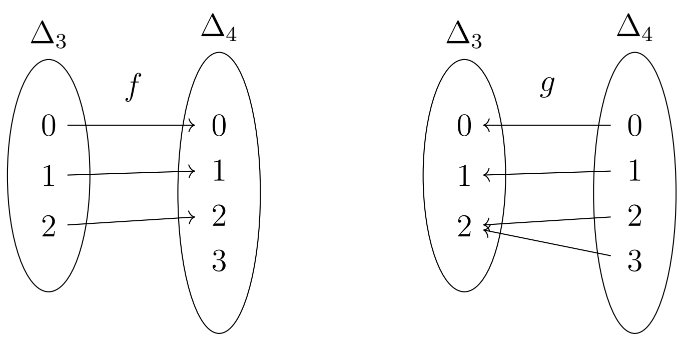


FIGURE 11. Two set maps $f : \{0,1,2\} \to \{0,1,2,3\}$ and $g : \{0,1,2,3\} \to \{0,1,2\}$.

For (2), apply a similar argument as in Part (1): since Δ_4 has more elements than Δ_3, any pointed tripod R satisfies

$$\frac{\pi}{2} = \frac{\pi}{2} - 0 \leqslant \mathrm{dis}(R).$$

It follows that $d_{\mathrm{GH}}\left(\left(\Delta_3, \mu^{(0)}_{\Delta_3}\right), \left(\Delta_4, \mu^{(0)}_{\Delta_4}\right)\right) \geqslant \frac{\pi}{4}$. To see that $d_{\mathrm{GH}}\left(\left(\Delta_3, \mu^{(0)}_{\Delta_3}\right), \left(\Delta_4, \mu^{(0)}_{\Delta_4}\right)\right) = \frac{\pi}{4}$, we utilize the same set maps f and g given in Figure 11 and construct the same tripod R as in (1). With the metrics $\mu^{(0)}_{\Delta_3}$ and $\mu^{(0)}_{\Delta_4}$, a direct calculation shows that $\mathrm{dis}(R) = \frac{\pi}{4}$.

For (3), we first apply Proposition 3.5 to see that

$$d_{\mathrm{GH}}\left(\left(L(\Delta_3), \mu^{(1)}_{\Delta_3}\right), \left(L(\Delta_4), \mu^{(1)}_{\Delta_4}\right)\right) \geqslant \frac{1}{2} \cdot \left(\pi - \frac{2\pi}{3}\right) = \frac{\pi}{6}.$$

Next we construct a tripod R whose distortion is $\frac{\pi}{3}$, by first defining two set maps f and g as in Figure 12. We set $Z = L(\Delta_3) \sqcup L(\Delta_4)$, and define set maps $\phi = (\mathrm{Id}_{L(\Delta_3)}, f)$ and $\psi = (g, \mathrm{Id}_{L(\Delta_4)})$. These define a pointed tripod R, with $\mathrm{dis}(R) = \frac{\pi}{3}$.

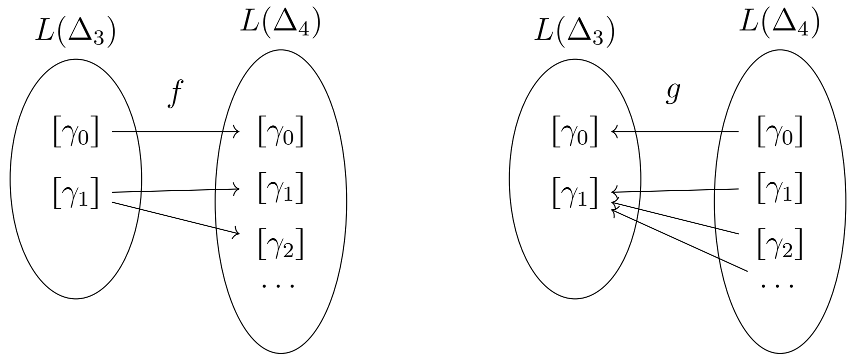


FIGURE 12. Two set maps $f : L(\Delta_3) \to L(\Delta_4)$ and $g : L(\Delta_4) \to L(\Delta_3)$. Recall from Page 51 that in the cycle graph C_n, we defined the discrete loop γ_r for $r = 1, \cdots, \left\lfloor \frac{n-1}{3} \right\rfloor$. As Δ_n and C_n have the same underlying space $\{0, \cdots, n-1\}$, the notation γ_r can be inherited. Although γ_r depends on n, we will not specify n in the notation when the concept is clear. The dots in $L(\Delta_4)$ represent $[\gamma_1^{*k}]$ for integer $k \neq 0, 1$.

For (4), because $(\Delta_n, d_n) \cong \left(C_n, \frac{2\pi}{n} d_{C_n}\right)$, we have

$$\mathrm{P}\Pi_1(\Delta_3) = \mathbb{0} \text{ and } \mathrm{P}\Pi_1(\Delta_4) = \mathbb{Z}\left[\tfrac{\pi}{2}, \pi\right).$$

Therefore, $\mathrm{dgm}_1(\Delta_3) = \varnothing$ and $\mathrm{dgm}_1(\Delta_4) = \{(\frac{\pi}{2}, \pi)\}$. It follows immediately that

$$d_{\mathrm{I}}(\mathrm{P}\Pi_1(\Delta_3), \mathrm{P}\Pi_1(\Delta_4)) = d_{\mathrm{I}}(\mathrm{PH}_1^{\mathrm{VR}}(\Delta_3), \mathrm{PH}_1^{\mathrm{VR}}(\Delta_4)) = d_{\mathrm{B}}(\mathrm{dgm}_1(\Delta_3), \mathrm{dgm}_1(\Delta_4)) = \tfrac{\pi}{4}.$$

□

Remark 7.14. The symmetry property of (Δ_n, d_n) guarantees that $d_{\mathrm{GH}}((\Delta_3, d_3), (\Delta_4, d_4)) = d_{\mathrm{GH}}^{\mathrm{pt}}((\Delta_3, 0, d_3), (\Delta_4, 0, d_4))$. In Theorem 4.3, Theorem 4.23 and Theorem 6.3, we have proved that (2), (3) and (4) are lower bounds of (1), the Gromov-Hausdorff distance, under certain restrictions.

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