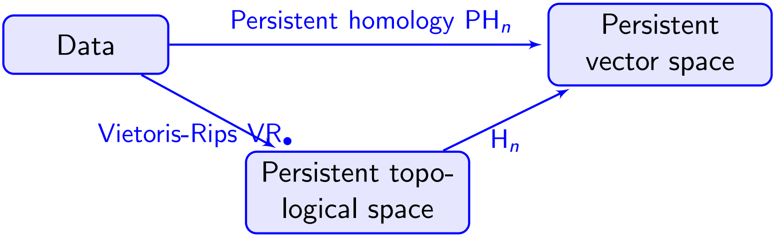
Persistent Homotopy Groups of Metric Spaces

Ling Zhou

Joint with: Facundo Mémoli

January 15, 2021


Data
Persistent homology PH_n
Persistent vector space
Vietoris-Rips $VR_\bullet$
H_n
Persistent topological space

Stability of Persistent Homology

- d_{GH}: the Gromov-Hausdorff distance of metric spaces, which is NP-hard to compute, [Schmiedl, 2017];
- d_{I}: the interleaving distance of persistent vector spaces, computable in polynomial time.

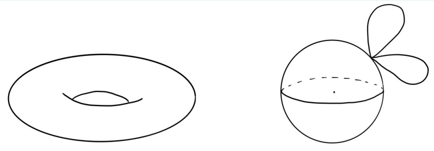
Theorem (Stability, [Chazal et al., 2014])

Let (X, d_X) and (Y, d_Y) be two metric spaces. Then, for any $n \in \mathbb{Z}_{\geq 0}$,

$$d_{\mathrm{I}}(\mathrm{PH}_n(X), \mathrm{PH}_n(Y)) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Homotopy Stronger than Homology

Example ($\mathbb{S}^1 \times \mathbb{S}^1$ vs. $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$)



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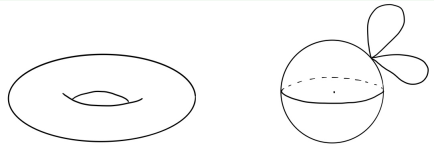

Figure 1: $\mathbb{S}^1 \times \mathbb{S}^1$ (left) and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ (right).

- $\mathsf{H}_n\left(\mathbb{S}^1 \times \mathbb{S}^1\right) \cong \mathsf{H}_n\left(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1\right)$, for any $n \in \mathbb{N}$.

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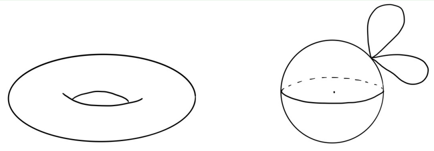

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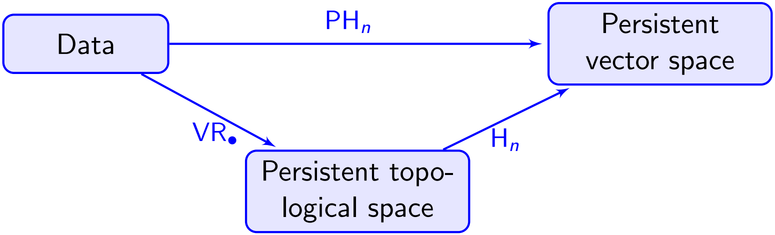
- $\pi_1\left(\mathbb{S}^1 \times \mathbb{S}^1\right) \cong \mathbb{Z} \times \mathbb{Z} \ncong \mathbb{Z} * \mathbb{Z} \cong \pi_1\left(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1\right)$.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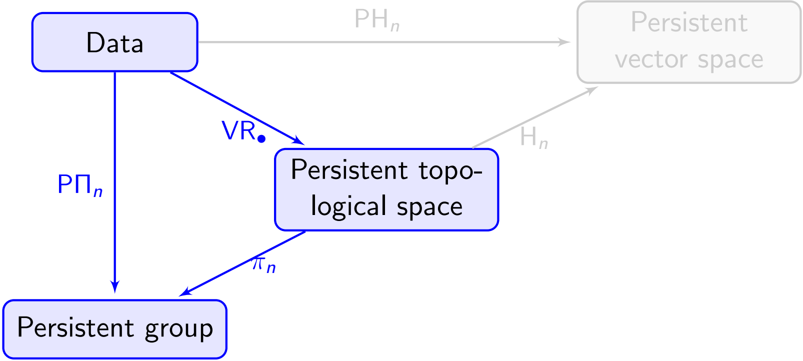
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π_n
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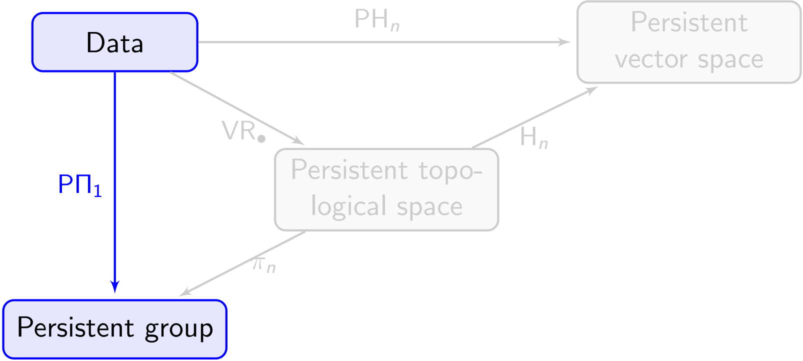
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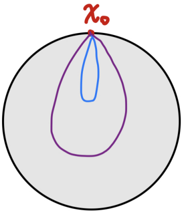
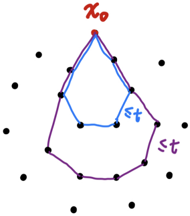
Discrete Fundamental Group





Definition (Discrete fundamental groups [Berestovskii, Plaut & Wilkins])

Let $\mathcal{L}^t(X, x_0) = \{t\text{-loops based at } x_0\}$. The **discrete fundamental group at scale** t is

$$\pi_1^t(X, x_0) := \mathcal{L}^t(X, x_0)/\sim_1^t .$$

Persistent Fundamental Group

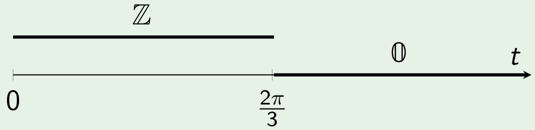
Definition (Persistent fundamental group)

The **persistent fundamental group** of (X, x_0) is the functor

$$
\begin{aligned}
\mathrm{P\Pi}_1^{\bullet}(X, x_0) : (\mathbb{R}_{>0}, \leq) &\to \mathsf{Grp} \\
t &\mapsto \pi_1^t(X, x_0) \\
(t \leq t') &\mapsto \left(\pi_1^t(X, x_0) \to \pi_1^{t'}(X, x_0)\right).
\end{aligned}
$$

Example

Let $\mathbb{S}^1$ be the unit circle. Then $\mathrm{P\Pi}_1(\mathbb{S}^1)$ is



Stability of $\mathrm{P}\Pi_n$

Theorem (Stability)

$$d_{\mathrm{I}}\left(\mathrm{P}\Pi_n(X), \mathrm{P}\Pi_n(Y)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Stability of $\mathrm{P\Pi}_n$

Theorem (Stability)

$$d_{\mathrm{I}}\left(\mathrm{PH}_1(X), \mathrm{PH}_1(Y)\right) \leq d_{\mathrm{I}}\left(\mathrm{P\Pi}_1(X), \mathrm{P\Pi}_1(Y)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Example ($\mathbb{S}^1 \times \mathbb{S}^1$ vs. $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$)

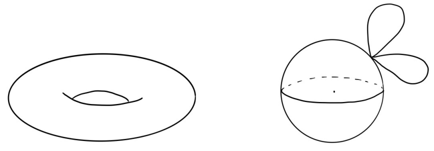


Figure 2: $\mathbb{S}^1 \times \mathbb{S}^1$ (left) and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ (right).

Let $a := \frac{1}{2}\arccos(-\frac{1}{3})$.

- $\mathrm{PH}_n|_{(0,2a)}\left(\mathbb{S}^1 \times \mathbb{S}^1\right) = \mathrm{PH}_n|_{(0,2a)}\left(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2\right), \forall n$;
- $\mathrm{P\Pi}_1|_{(0,2a)}(\mathbb{S}^1 \times \mathbb{S}^1) \ncong \mathrm{P\Pi}_1|_{(0,2a)}(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2)$.

And the stability of $\mathrm{P\Pi}_1$ implies

$$0.96 \approx \frac{1}{2} \cdot a \leq d_{\mathrm{GH}}\left(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2, \mathbb{S}^1 \times \mathbb{S}^1\right).$$

Dendrogram and Metric on $\pi_1(X)$

Theorem ([Mémoli and Zhou, 2019])

Let X satisfy some fairly mild assumptions (geodesic and semi-locally simply connected). Associated to $\mathrm{P\Pi}_1(X)$, there is a dendrogram $\theta_{\pi_1(X)}$ over $\pi_1(X)$ given by

$$\theta_{\pi_1(X)}(t) := \pi_1^t(X), \forall t > 0.$$

In addition, the dendrogram induces an ultrametric $\mu_{\theta_{\pi_1(X)}}$ on $\pi_1(X)$.

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Example

Let Y be simply-connected, i.e. $\pi_1(Y) = 0$. Then $\mathrm{P}\Pi_1(Y) = \mathbb{0}$.

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Theorem (d_{GH}-stability for $\theta_{\pi_1(\bullet)}$)

If compact geodesic metric spaces X and Y are s.l.s.c., then

$$d_{\mathrm{GH}}\left(\left(\pi_1(X), \mu_{\theta_{\pi_1(X)}}\right), \left(\pi_1(Y), \mu_{\theta_{\pi_1(Y)}}\right)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Example (Dendrogram over $\pi_1(\mathbb{S}^1)$)

Associated to $\mathsf{P\Pi}_1(\mathbb{S}^1)$ we have a dendrogram over $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$:

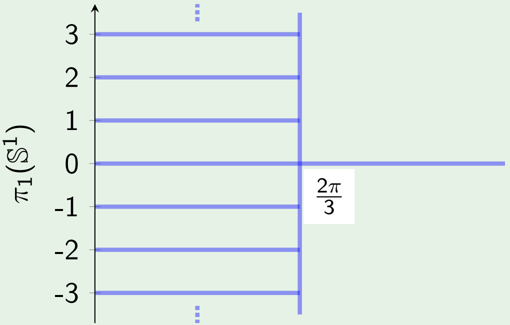


Figure 3: The *y*-axis represents elements of $\pi_1(\mathbb{S}^1) = \mathbb{Z}\gamma$, for γ a generator of $\mathbb{S}^1$.

Example (Dendrogram over $\pi_1(\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2))$)

For $0 < r_1 \leq r_2$, associated to $\mathrm{P}\Pi_1\left(\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)\right)$ we have a dendrogram:

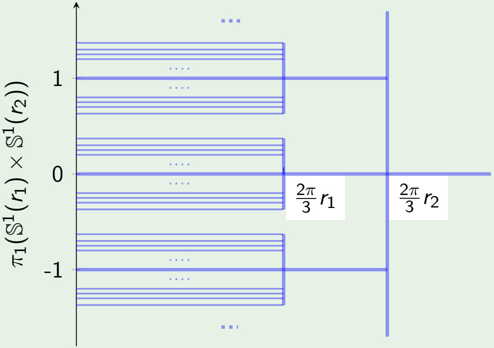
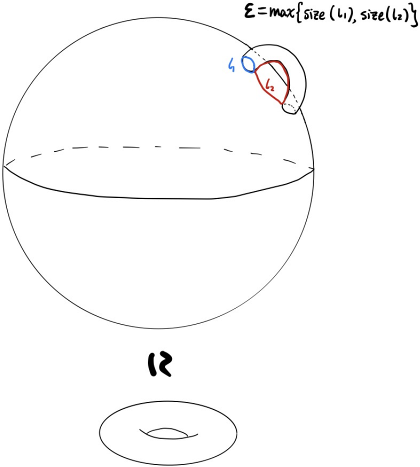
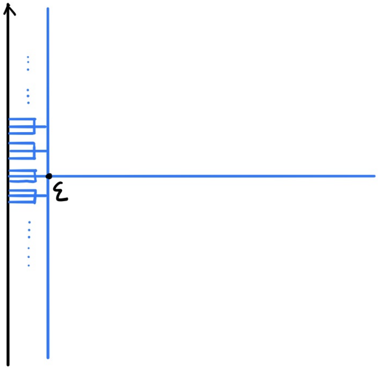


Figure 4: The y-axis represents elements of $\mathbb{Z}\gamma_1 \times \mathbb{Z}\gamma_2$, where γ_1 and γ_2 are generators of $\pi_1(\mathbb{S}^1(r_1))$ and $\pi_1(\mathbb{S}^1(r_2))$, respectively.

[Gromov, 1999]: How simply-connected is a space?





Persistent Rational Homotopy

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Rational homotopy groups of spheres ([Serre, 1951]):

$$\pi_n(\mathbb{S}^{2k-1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & n = 2k-1, \\ 0, & \text{otherwise}, \end{cases}$$

the same as $\mathrm{H}_n(\mathbb{S}^{2k-1})$, and

$$\pi_n(\mathbb{S}^{2k}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & n = 2k, 4k-1, \\ 0. & \text{otherwise}. \end{cases}$$

Persistent Rational Homotopy

Define $\mathsf{P}\Pi_n(X) \otimes \mathbb{Q}$ to be the composition:

$$(\mathbb{R}_{>0}, \leq) \xrightarrow{\mathsf{P}\Pi_n(X)} \mathsf{Ab\,Grp} \xrightarrow{-\otimes\mathbb{Q}} \mathsf{Vec}\,.$$

Persistent Rational Homotopy

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With the computation of the homotopy type of $\mathrm{VR}_{\leq r}(\mathbb{S}^1)$ from [Adamaszek and Adams, 2017], we obtain

Example

If $n = 4k-1$ for some $k \in \mathbb{N}$,

$$\mathrm{P\Pi}^{\leq}_{4k-1}\left(\mathbb{S}^1\right) \otimes \mathbb{Q} \cong \mathrm{PH}^{\leq}_{4k-1}\left(\mathbb{S}^1\right) \oplus \mathbb{Q}^{\times\infty}\left[\tfrac{2k}{4k+1}, \tfrac{2k}{4k+1}\right].$$

Otherwise, $\mathrm{P\Pi}^{\leq}_{n}\left(\mathbb{S}^1\right) \otimes \mathbb{Q} \cong \mathrm{PH}^{\leq}_{n}\left(\mathbb{S}^1\right)$.

Proposition

Let X and Y be compact metric spaces. Then for each $n \in \mathbb{Z}_{\geq 2}$,

$$d_{\mathrm{I}}\left(\mathsf{P\Pi}_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathsf{P\Pi}_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

When $\mathsf{P\Pi}_1(X), \mathsf{P\Pi}_1(Y) \in \mathsf{PAb}$, *we also have*

$$d_{\mathrm{I}}\left(\mathsf{P\Pi}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathsf{P\Pi}_1(Y) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Future Work

- Algorithms to compute persistent rational homotopy groups, using Sullivan minimal models, [Peterson, 2015];

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- Construct $P\Pi_n$ using discrete homotopy groups;
- Compute persistent fundamental groups with some restrictions on the data, [Brendel et al., 2015].

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Homologie singulière des espaces fibrés.
Ann. of Math, 54(2):425–505.

Thank You!

Let $t > 0$. The **Vietoris–Rips complex** $\mathrm{VR}_t(X)$ of X is the simplicial complex with vertex set X, where

$$\text{a finite subset } \sigma \subset X \text{ is a face of } \mathrm{VR}_t(X) \Leftrightarrow \mathrm{diam}(\sigma) < t.$$

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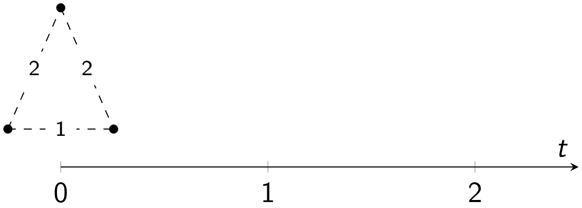

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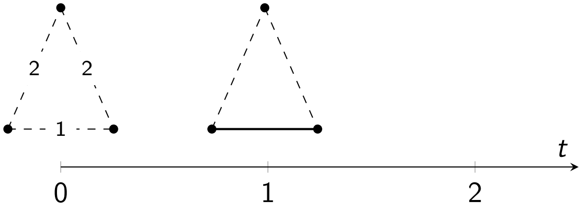

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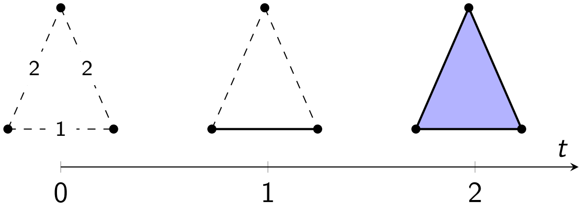

Example ($\mathbb{S}^1$)

From [Adamaszek and Adams, 2017]:

$$\mathrm{VR}_{<\epsilon}(\mathbb{S}^1) \simeq \mathbb{S}^{2k+1}, \text{ if } \epsilon \in \left(\frac{k}{2k+1}, \frac{k+1}{2k+3}\right] \text{ for some } k \in \mathbb{N}.$$

For $\frac{k}{2k+1} < \epsilon \leq \epsilon' \leq \frac{k+1}{2k+3}$, $\mathrm{VR}_{<\epsilon}(\mathbb{S}^1) \overset{\simeq}{\hookrightarrow} \mathrm{VR}_{<\epsilon'}(\mathbb{S}^1)$.

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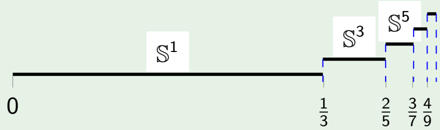

- $\mathrm{PH}_n^{\mathrm{VR}}(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z}\left(\frac{k-1}{2k-1}, \frac{k}{2k+1}\right], & \text{if } n = 2k-1, \\ 0, & \text{if } n = 2k. \end{cases}$

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- $\mathrm{P\Pi}_1^{\mathrm{VR}}(\mathbb{S}^1) \cong \mathbb{Z}\left(0, \frac{1}{3}\right] \cong \mathrm{P\Pi}_1(\mathbb{S}^1)$ and $\mathrm{P\Pi}_3^{\mathrm{VR}}(\mathbb{S}^1) \cong \mathbb{Z}\left(\frac{1}{3}, \frac{2}{5}\right]$.

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- $\mathrm{P\Pi}_1^{\mathrm{VR}}(\mathbb{S}^1) \cong \mathbb{Z}\left(0, \frac{1}{3}\right] \cong \mathrm{P\Pi}_1(\mathbb{S}^1)$ and $\mathrm{P\Pi}_3^{\mathrm{VR}}(\mathbb{S}^1) \cong \mathbb{Z}\left(\frac{1}{3}, \frac{2}{5}\right]$.
- Question: how to compute $\mathrm{P\Pi}_n^{\mathrm{VR}}(\mathbb{S}^1)$ for n large?