Homotopical decompositions of simplicial and Vietoris-Rips complexes

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From local to global

Homology of the components

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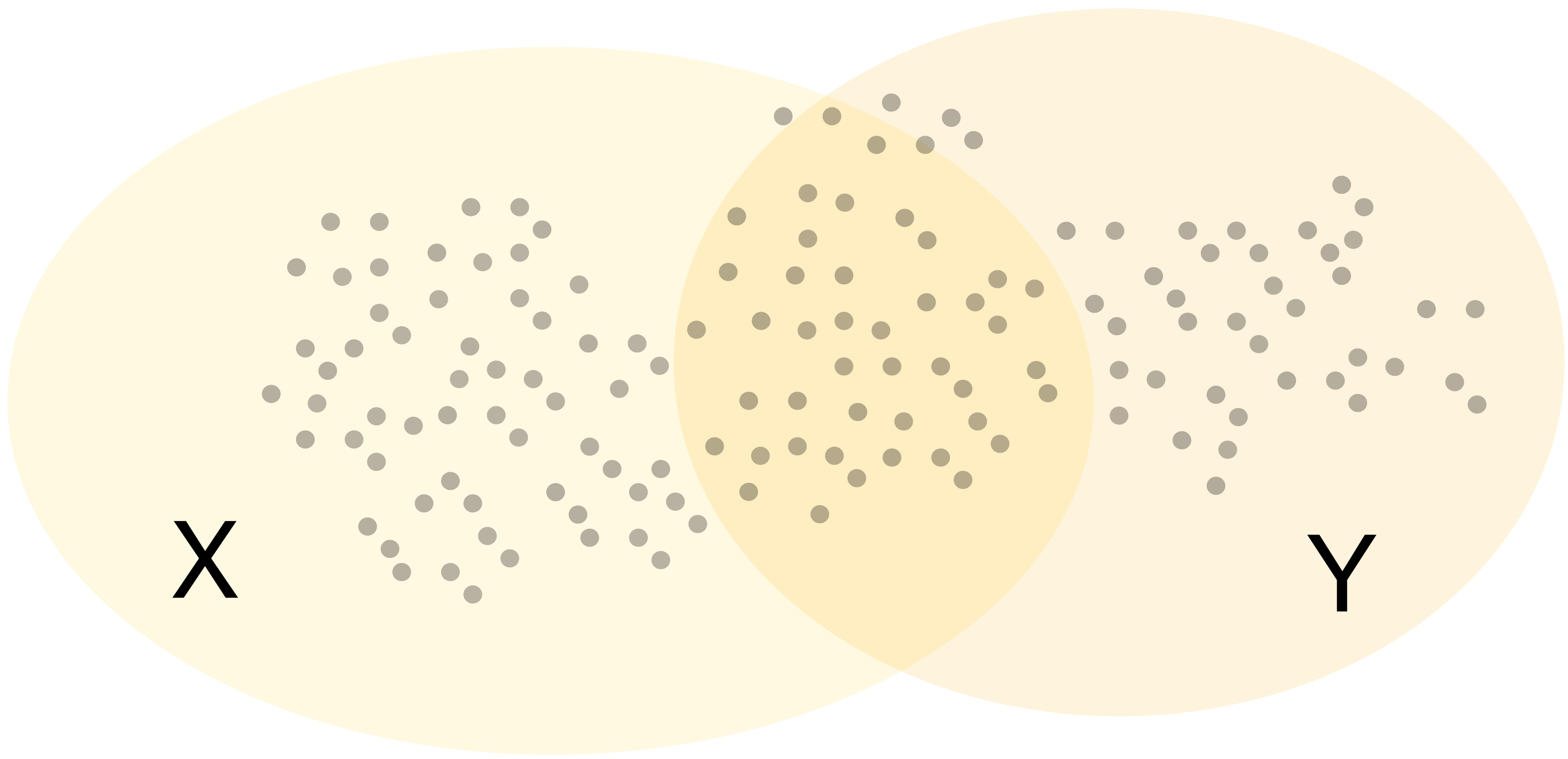
Homology of the simplicial complex



$$\mathrm{VR}_r(X) \cup \mathrm{VR}_r(Y) \subset \mathrm{VR}_r(X \cup Y)$$

In general this inclusion does not lead to homology isomorphism

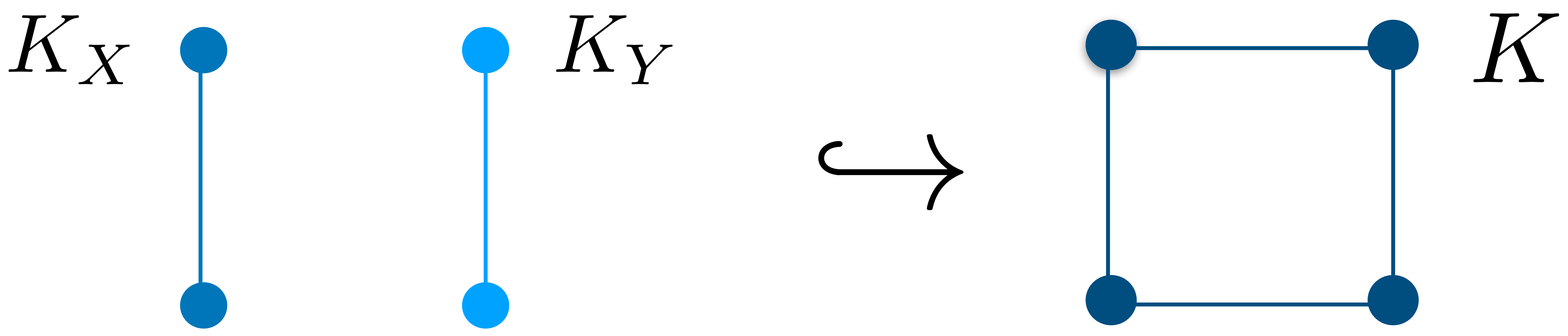
The problem

Let K be a simplicial complex and $K_0 = X \cup Y$ a covering of its vertices

$$K_X \cup K_Y \hookrightarrow K$$

Data-driven decomposition

Can we measure their difference?

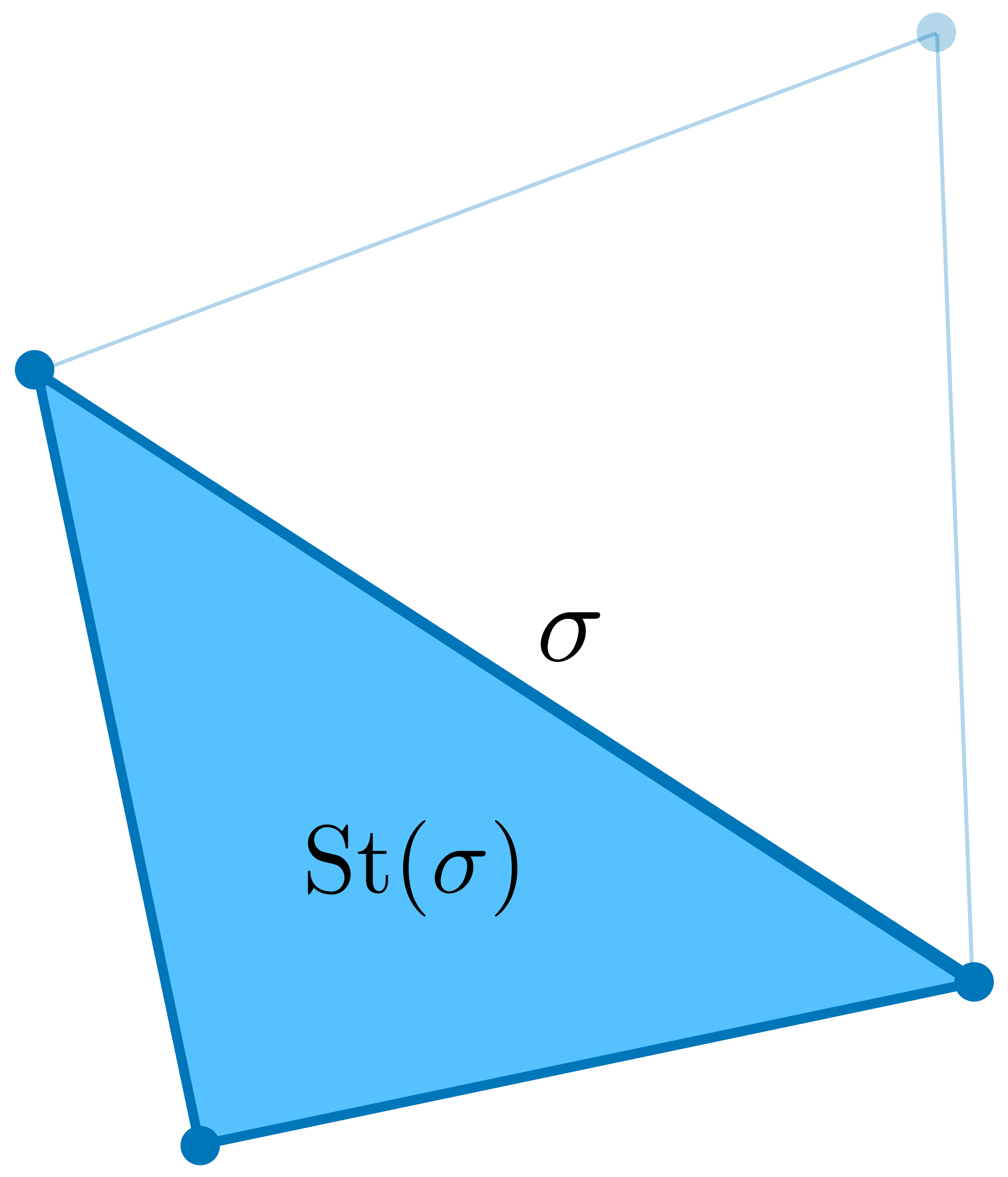


Star

The star of a simplex σ in K is the **subcomplex**

$$\mathrm{St}(\sigma) := \{\mu \in K \mid \sigma \cup \mu \in K\}$$

It is contractible for every σ.



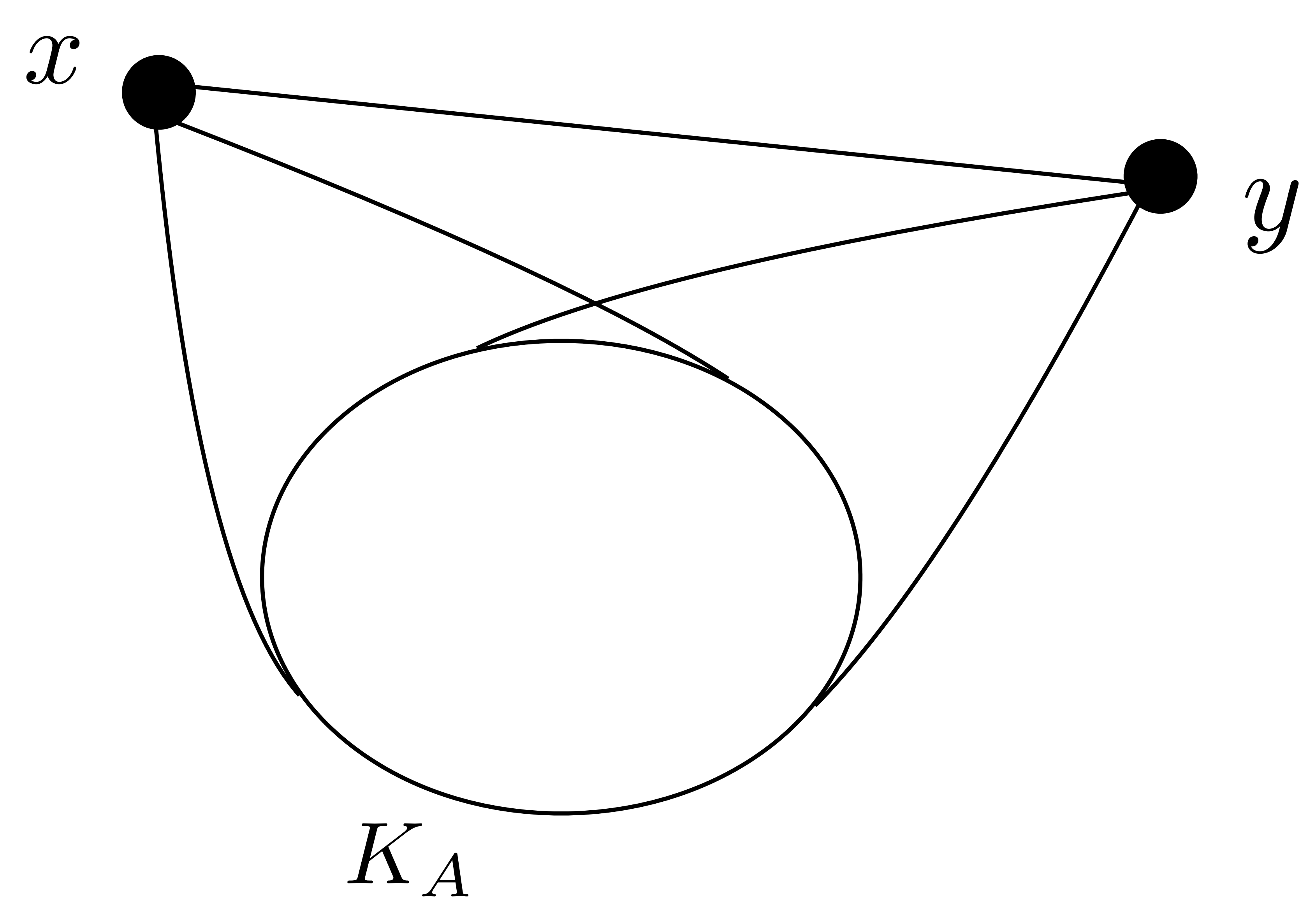
2 points outside

Let $X = K_0 \setminus \{y\}$, $Y = K_0 \setminus \{x\}$ and $A = X \cap Y = K_0 \setminus \{x, y\}$.

If $\{x, y\}$ is a simplex in K we can define $\mathrm{St} := K_A \cap \mathrm{St}(\{x, y\})$

leading to a cofibration sequence

$$\Sigma \mathrm{St} \hookrightarrow K_X \cup K_Y \hookrightarrow K$$



it is a simplicial complex

In general, if $K_0 = X \cup Y$, $A = X \cap Y$ and σ simplex in K,

$$\mathrm{St}(\sigma, A) = \{\mu \subset A \mid |\mu| > 0 \text{ and } \mu \cup \sigma \in K\} = K_A \cap \mathrm{St}(\sigma)$$

is the **obstruction complex** of σ.

it might be empty

n+1 points outside

Let σ be a subset of $n+1$ distinct vertices of K.

- If σ does not form a simplex in K, then $\bigcup_{v\in\sigma} K_{K_0\setminus\{v\}} = K$.
- If σ forms a simplex in K, then there is a cofibration sequence:

$$\Sigma^n \mathrm{St}(\sigma, K_0 \setminus \sigma) \to \bigcup_{v\in\sigma} K_{K_0\setminus\{v\}} \hookrightarrow K$$

$\mathrm{St}(\sigma, K_0 \setminus \sigma) \neq \emptyset$ and $\bar{H}_i(\mathrm{St}(\sigma, K_0 \setminus \sigma), \mathbb{Z}) = 0$ for all i

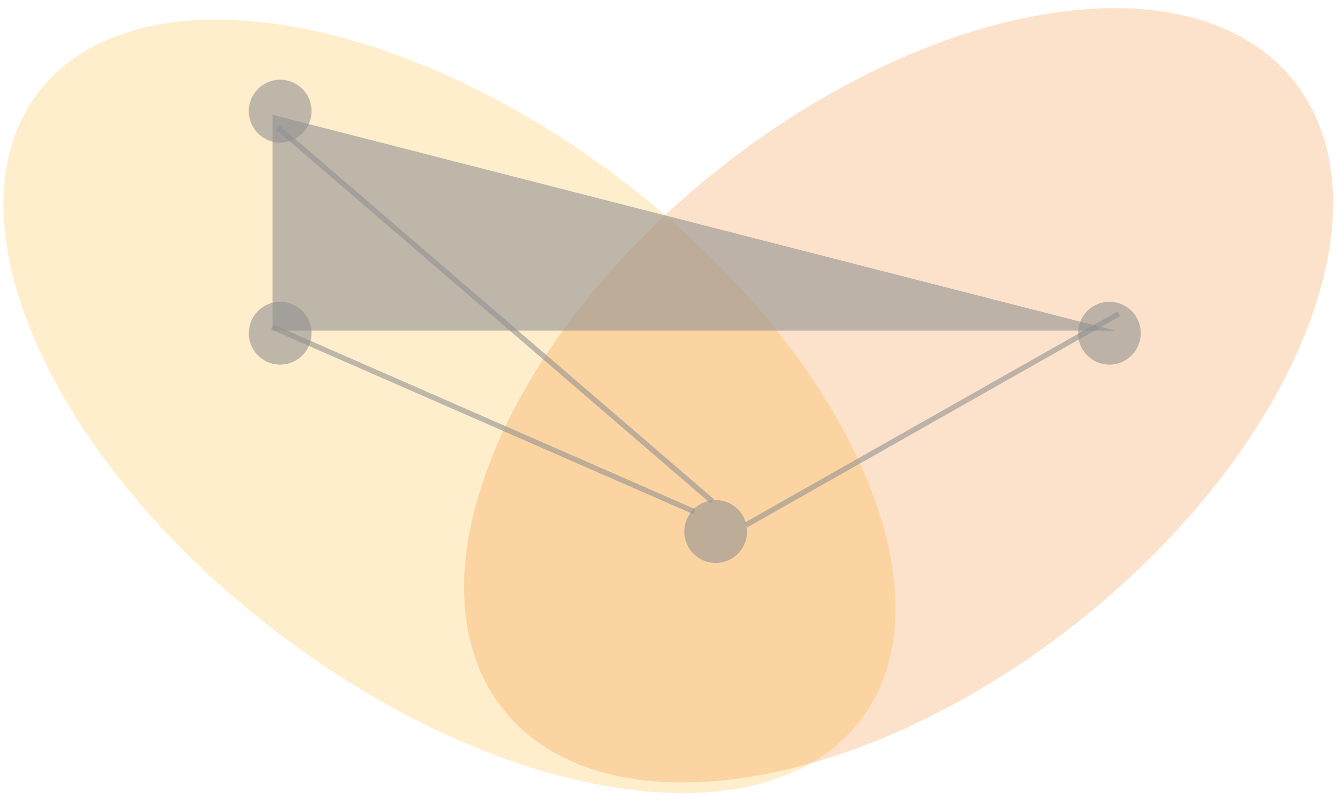
$\updownarrow$

$\bigcup_{v\in\sigma} K_{K_0\setminus\{v\}} \hookrightarrow K$ is a weak equivalence

For example, if $\mathrm{St}(\sigma, K_0 \setminus \sigma)$ is contractible.

Decompositions I

$\sigma \in K \setminus P$



Let P be the subposet of K given by

$$P := \{\sigma \in K \mid \sigma \subset X \text{ or } \sigma \subset Y \text{ or } \sigma \cap A \neq \emptyset\}$$

and hence the set of simplices that are not in P is

$$K \setminus P = \{\sigma \in K \mid \sigma \cap X \neq \emptyset \text{ and } \sigma \cap Y \neq \emptyset \text{ and } \sigma \cap A = \emptyset\}$$

$$K_X \cup K_Y \hookrightarrow P \hookrightarrow K$$

Always a weak equivalence

Its fibers are weakly equivalent to the obstruction complexes

Decompositions II

Let $\mathcal{C}$ be a closed collection of simplicial sets. Assume that, for every σ in $K \setminus P$, the obstruction complex $\mathrm{St}(\sigma, A)$ satisfies $\mathcal{C}$. Then the homotopy fibers of the inclusion $K_X \cup K_Y \subset K$ also satisfy $\mathcal{C}$.

Explicit examples of closed collections

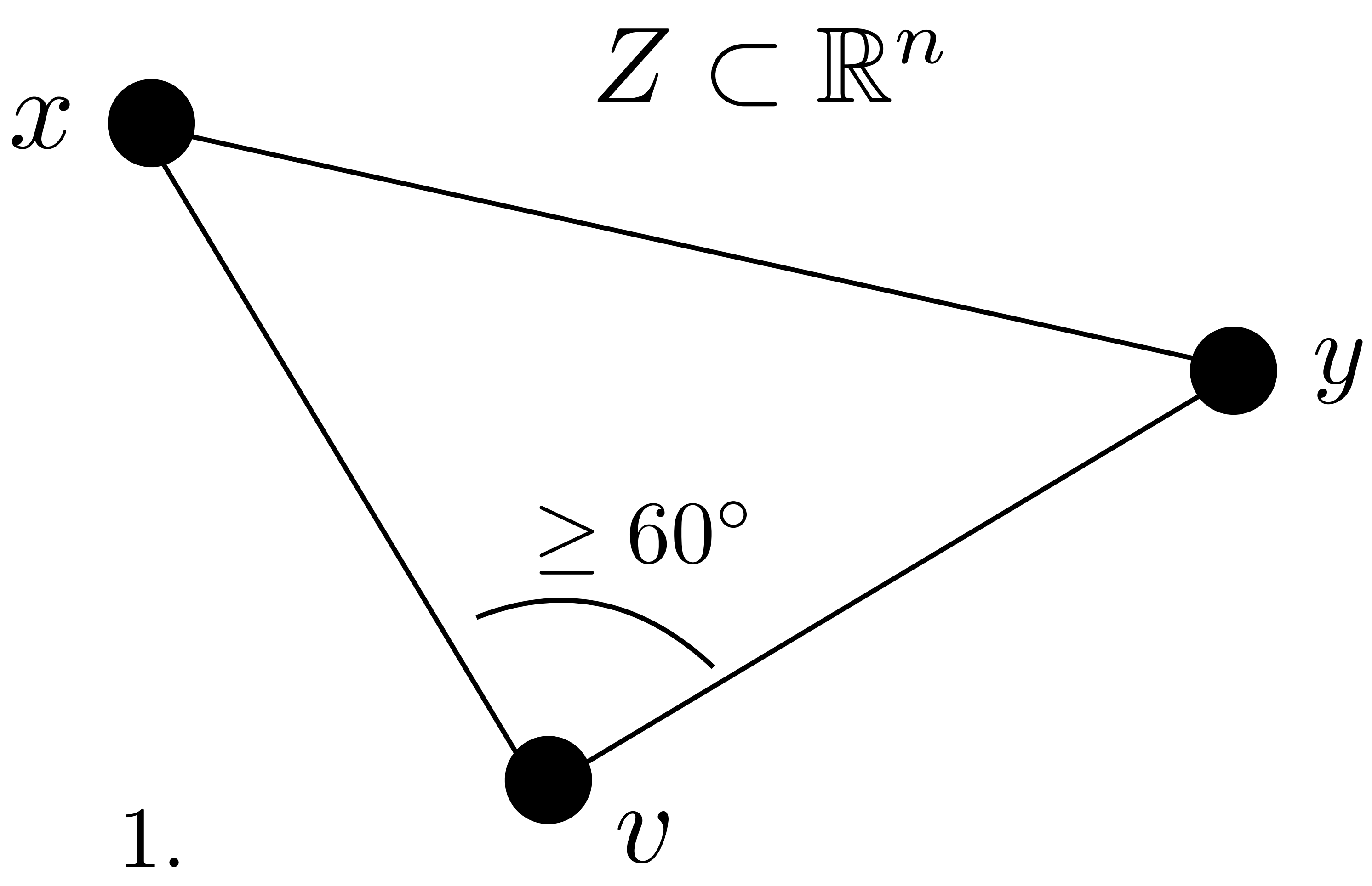
- If, for every σ in $K \setminus P$, the simplicial complex $\mathrm{St}(\sigma, A)$ is contractible, then $K_X \cup K_Y \subset K$ is a weak equivalence.
- If, for every σ in $K \setminus P$, the simplicial complex $\mathrm{St}(\sigma, A)$ is $n-$connected, then the homotopy fibers of $K_X \cup K_Y \subset K$ are $n-$connected and this inclusion induces an isomorphism on homotopy groups in degrees $0, \ldots, n$ and a surjection in degree $n+1$.

Vietoris-Rips for distances

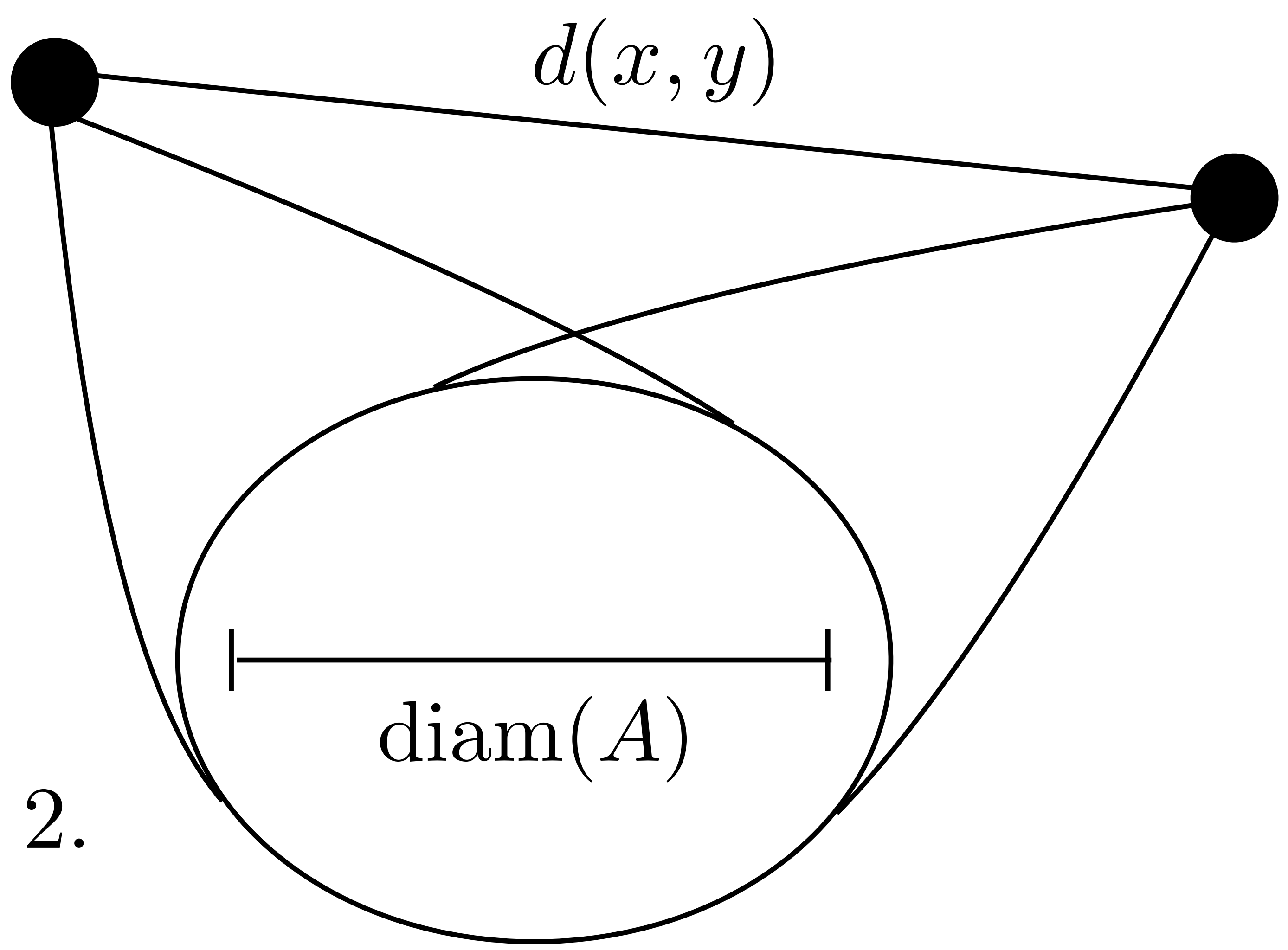
Let (Z, d) be a distance space, $X \cup Y = Z$ be a covering of Z, and $A = X \cap Y$ be non-empty. Assume that for every x in $X \setminus A$, y in $Y \setminus A$ and v in A,

1. $d(x, y) \geq d(x, v)$ and $d(x, y) \geq d(y, v)$
2. $d(x, y) \geq \mathrm{diam}(A)$

Then $\mathrm{VR}_r(X) \cup \mathrm{VR}_r(Y) \hookrightarrow \mathrm{VR}_r(Z)$ is a weak equivalence for all r in $[0, \infty)$.

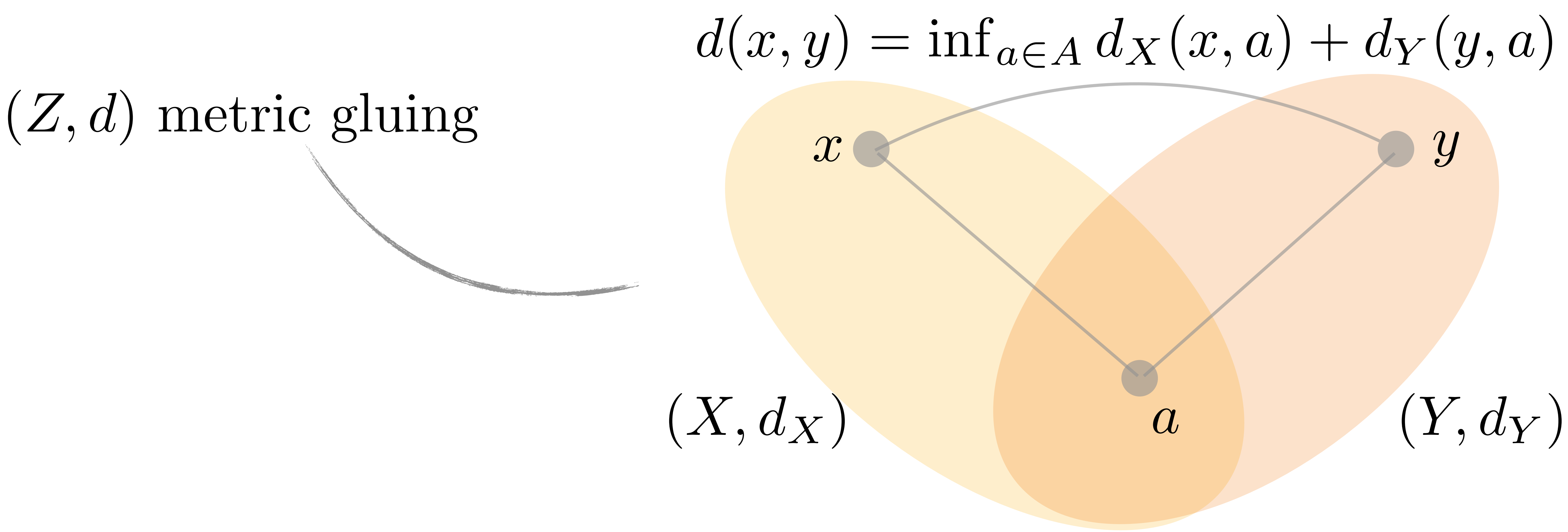


1.



2.

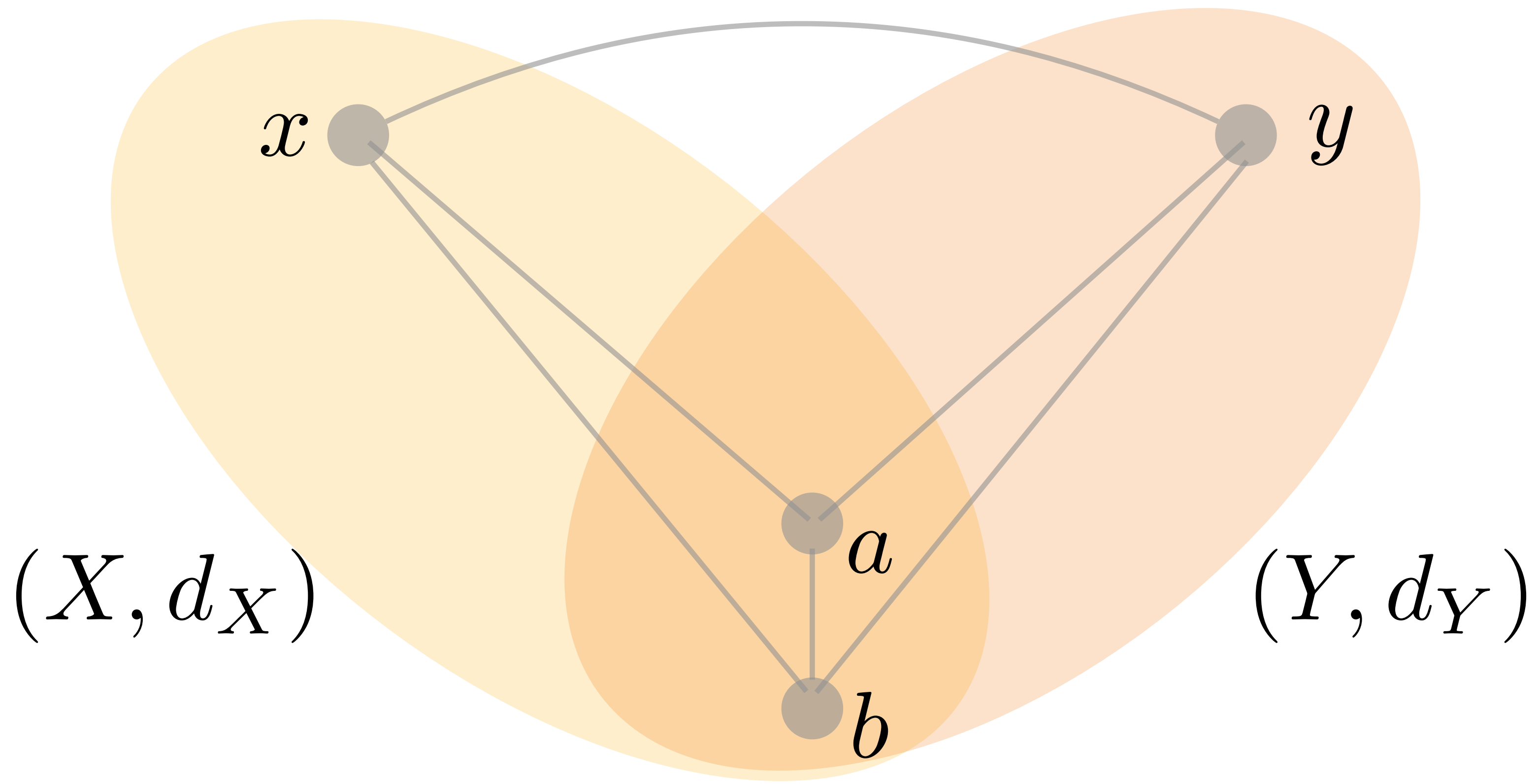
Vietoris-Rips for pseudometrics



Assume that for every vertex v in an edge σ in $\mathrm{VR}_r(Z) \setminus P$, if a and b are elements in A such that $d(a,v) \leq r$ and $d(v,b) \leq r$, then $2d(a,b) \leq d(a,v) + d(v,b)$.

Adamaszek, M., Adams, H., Gasparovic, E., Gommel, M., Purvine, E., Sazdanovic, R., Wang, B., Wang, Y., Ziegelmeier, L.: On homotopy types of vietoris-rips complexes of metric gluings. arXiv: 1712.06224 (2020). URL https://arxiv.org/abs/1712.06224

Vietoris-Rips for pseudometrics



For the first time triangular inequality plays a role

Assume that for every vertex v in an edge σ in $\mathrm{VR}_r(Z) \setminus P$, if a and b are elements in A such that $d(a, v) \leq r$ and $d(v, b) \leq r$, then $2d(a, b) \leq d(a, v) + d(v, b)$. Then the homotopy fibers of the inclusion $\mathrm{VR}_r(X) \cup \mathrm{VR}_r(Y) \subset \mathrm{VR}_r(Z)$ are simply connected and this map induces an isomorphism on π_0 and π_1 and a surjection on π_2.