Computing the Homology of Semialgebraic Sets via TDA

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Semialgebraic Sets

Formed from

atomic semialgebraic sets

i.e. sets described by

$(f=0), (f>0), (f\geq 0), (f<0), (f\leq 0)$

with f real polynomial

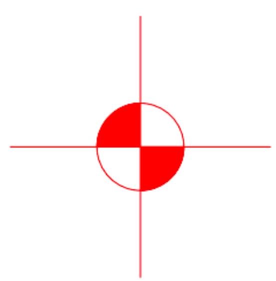
using **set-theoretical operations**

i.e. $\wedge$: intersection

$\vee$: union

$\neg$: complement

Example:



$((XY \leq 0) \wedge ((X^2+Y^2 \leq 1) \vee (XY=0))) \vee (X^2+Y^2=1)$

Why do we care?

Natural descriptions of many things are semialgebraic sets

Topological hardness

- Every finite simplicial complex is a semialgebraic set.
- (Gabrielov, Vorobjov; 2005, 2009)

$$\beta(S) \leq \mathcal{O}(q^2 D)^n$$

Main Result

THM There is a numerically stable algorithm that, given $f \in \mathbb{R}[X_1,\dots,X_n]^q$ with $\deg f \leq D$ and a semialg. formula Φ of size $\leq s$, computes $H_0(S(f,\Phi)), \dots, H_n(S(f,\Phi))$, where $S(f,\Phi)$ is the semialg. set described by f and Φ, in $s(qD)^{\mathcal{O}(n^3)}$-time with probability $\geq 1-(2qD)^{-n}$ when f is 'random'.

Outline of the algorithm

0) Homogeneization of f
1) Estimation cond. number of f, $\bar{\kappa}(f)$
2) Gabrielov-Vorobjov construction (general ineq → lax ineq) [Hard to make explicit!]
3) Create uniform grid for sample
4) Simplicial reconstruction of $S(f,\Phi)$: Construct simplicial model of $S(f,\Phi)$ by using Φ and simplicial model of atoms

Remarks

- All other algorithms have doubly exponential complexity in n
- Numerical algorithms ⇒ { input-dependent run-time; possible ill-posed inputs; can handle errors }
- Main TDA tool: Niyogi-Smale-Weinberger Thm
- (Cucker, Krick, Shub; 2018) Hardness of sampling dominated by cond. number of f in the algebraic case.
  (Bürgisser, Cucker, Lairez; 2018) Ext. to basic semialg. sets (no unions!)
- Unions require simplicial reconstruction!

Simplicial Reconstruction

Key observation (Non-formal)

- $\mathcal{X}_i$: sample of X_i;

IF

$\forall I$, $\bigcap_{i\in I} \mathcal{X}_i$ 'good' sample of $\bigcap_{i\in I} X_i$

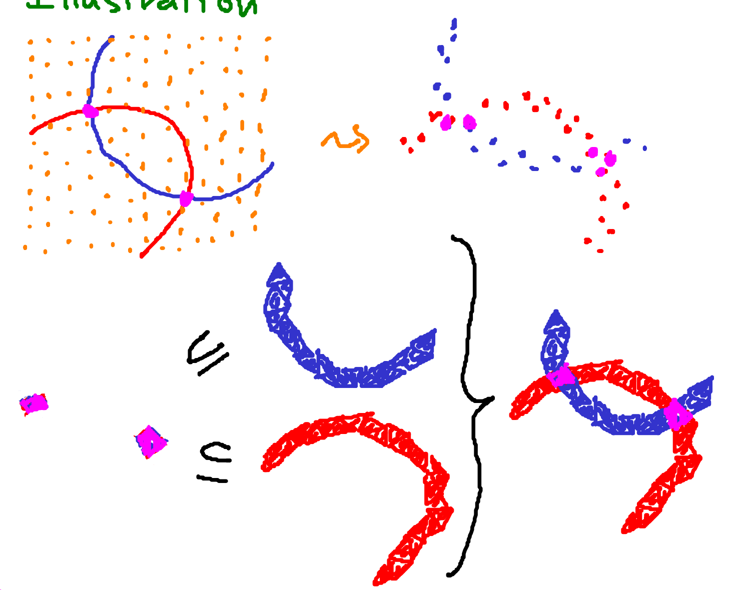
THEN

$\bigcup \check{C}_\varepsilon(\mathcal{X}_i)$ and $\bigcup X_i$ same homology

Remarks

- Proof uses a form of Vietoris-Begle with $\pi: \bigcup \check{C}_\varepsilon(\mathcal{X}_i) \to \bigcup\{B_\varepsilon(y) \mid y \in \bigcup \mathcal{X}_i\}$
- Valid also for Vietoris-Rips complex

Illustration



References

P. Bürgisser, F. Cucker, J. Tonelli-Cueto. Computing the Homology of Semialgebraic Sets. I: Lax Formulas & II. General Formulas.
J. Tonelli-Cueto. Condition and Homology in Semialgebraic Geometry.